Modular representations, old and new

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G is a finite group.

The character of a representation of G over an algebraically closed field of characteristic 0 is an "ordinary" character. The set of ordinary characters of G is denoted by $\mathrm{Irr}(G)$.

Frobenius computed the character table of $PSL(2, p)$ in 1896.

Frobenius induction takes characters of a subgroup H of G to characters of G.

Richard Brauer developed the modular representation theory of finite groups, starting in the thirties.

G	a finite group
p	a prime integer
K	a sufficiently large field of characteristic 0
$\mathcal{O}$	a complete discrete valuation ring with quotient field K
k	residue field of $\mathcal{O}$, char k=p

A representation of G over K is equivalent to a representation over $\mathcal{O}$, and can then be reduced mod p to get a modular representation of G over k.

Brauer defined the character of a modular representation: a complex-valued function on the p-regular elements of G. Then we can compare ordinary and p-modular (Brauer) characters.

The decomposition map $d : K_0(KG) \to K_0(kG)$, where K_0 denotes the Grothendieck group, basis indexed by simple modules, expresses an ordinary character in terms of Brauer characters by going mod p.

The decomposition matrix D (over $\mathbf{Z}$) is the transition matrix between ordinary and Brauer characters. Entries of D are decomposition numbers.

The algebra KG is semisimple, but in the cases of interest, kG is not.

$$kG = B_1 \oplus B_2 \oplus \ldots \oplus B_n$$

where the B_i are "block algebras", indecomposable ideals of kG.

Leads to:

- a partition of the ordinary characters, or KG-modules, into blocks
- a partition of the Brauer characters, or kG-modules, into blocks
- a partition of the decomposition matrix into blocks

Example: A_5, $p = 2$: Ordinary characters in "Principal Block"

order of element	1	2	3	3	5
classsize	1	15	20	12	12
χ_1	1	1	1	1	1
χ_2	5	1	-1	0	0
χ_3	3	-1	0	$\frac{1-\sqrt{5}}{2} - 1$	$\frac{1+\sqrt{5}}{2} - 1$
χ_4	3	-1	0	$\frac{1+\sqrt{5}}{2} - 1$	$\frac{1-\sqrt{5}}{2} - 1$

Example: A_5, $p = 2$: Brauer characters in "Principal Block"

order of element	1	3	3	5
classsize	1	20	12	12
ψ_1	1	1	1	1
ψ_2	2	-1	$\frac{1+\sqrt{5}}{2} - 1$	$\frac{1-\sqrt{5}}{2} - 1$
ψ_3	2	-1	$\frac{1-\sqrt{5}}{2} - 1$	$\frac{1+\sqrt{5}}{2} - 1$

Decomposition matrix for Principal Block of A_5: $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

Example: $G = S_n$. If $\chi \in \mathrm{Irr}(G)$ then $\chi = \chi_\lambda$ where λ is a partition of n. Then there is a Young diagram corresponding to λ and p-hooks, p-cores are defined. Then:

Theorem (Brauer-Nakayama) χ_λ, χ_μ are in the same p-block if and only if λ, μ have the same p-core.

Example: Removing 3-hooks to get a 3-core:

$$\begin{pmatrix} * & * & * & * \\ * & * & * & \\ * & + & + & \\ * & + & & \end{pmatrix} \rightarrow \begin{pmatrix} * & * & + & + \\ * & * & + & \\ * & & & \\ * & & & \end{pmatrix} \rightarrow \begin{pmatrix} * & * & & \\ * & * & & \\ * & & & \\ * & & & \end{pmatrix}$$

Some main problems of modular representation theory:

- Describe the irreducible modular representations, e.g. their degrees
- Describe the blocks
- Find the decomposition matrix D

$\mathbf{G}$ is a connected reductive algebraic group defined over $\mathbf{F}_q$,

$F : \mathbf{G} \to \mathbf{G}$ a Frobenius morphism,

$G = \mathbf{G}^F$ is a finite reductive group.

Examples: $GL(n, q)$, $U(n, q)$, $Sp(2n, q)$, $SO^{\pm}(2n, q)$

For $GL(n, q)$, $F : (x_{ij}) \to (x_{ij}^q)$.

G has subgroups maximal tori, Levi subgroups (centralizers of tori), parabolic subgroups

$G = GL(n, q)$ has subgroups:

- Tori, abelian subgroups (e.g. diagonal matrices)
- Levi subgroups, products of subgroups of the form $GL(m, q^d)$
- Borel subgroups, isomorphic to "upper triangular matrices"
- Parabolic subgroups of the form $P = LV$, L a product of subgroups of the form $GL(m, q)$, $V \lhd P$

Parabolic subgroup P is of the form

$$\begin{pmatrix} \clubsuit & * & * & \dots & * \\ 0 & \clubsuit & * & \dots * & \\ . & . & . & . & . \\ 0 & 0 & 0 & 0 & \clubsuit \end{pmatrix}$$

Then L is of the form

$$\begin{pmatrix} \clubsuit & 0 & 0 & \dots & 0 \\ 0 & \clubsuit & 0 & \dots 0 & \\ . & . & . & . & . \\ 0 & 0 & 0 & 0 & \clubsuit \end{pmatrix}$$

And V is of the form $\begin{pmatrix} I & * & * & \dots & * \\ 0 & I & * & \dots * & \\ . & . & . & . & . \\ 0 & 0 & 0 & 0 & I \end{pmatrix}$

Let $P = LV$ as before.

Harish-Chandra induction is the following map:

$$R_L^G : K_0(KL) \to K_0(KG).$$

If $\psi \in \mathrm{Irr}(L)$ then $R_L^G(\psi) = \mathrm{Ind}_P^G(\tilde{\psi})$ where $\tilde{\psi}$ is the character of P obtained by inflating ψ to P.

Now let ℓ be a prime not dividing q.

Suppose L is a Levi subgroup, not necessarily in a parabolic subgroup P of G.

The Deligne-Lusztig linear operator:

$$R_L^G : K_0(\overline{\mathbf{Q}}_l L) \to K_0(\overline{\mathbf{Q}}_l G).$$

If $L \leqslant P \leqslant G$, where P is a parabolic subgroup, R_L^G is just Harish-Chandra induction.

Example: $G = GL(n, q)$. If L is the subgroup of diagonal matrices contained in the (Borel) subgroup B of upper triangular matrices, we can do Harish-Chandra induction. But if L is a torus (Coxeter torus) of order $q^n - 1$, we must do Deligne-Lusztig induction to obtain generalized characters from characters of L.

G is a finite reductive group.

A unipotent character is a constituent of $R_T^G(1)$.

Example:

Let $G = GL(n, q)$. The unipotent characters of G are constituents of Ind_B^G and are indexed by partitions of n. Denoted by χ_λ, λ a partition of n.

ℓ a prime not dividing q, e the order of q mod ℓ

Theorem (Fong-Srinivasan) χ_λ, χ_μ are in the same ℓ-block if and only if λ, μ have the same e-core.

Example: $n = 5$, ℓ divides $q + 1$, $e = 2$. Then χ_λ for $5,\ 32,\ 31^2,\ 2^2 1,\ 1^5$ are in a block. Same for S_5, $p = 2$.

Example: $n = 4$, ℓ divides $q^2 + 1$, $e = 4$. Then χ_λ for $4,\ 31,\ 21^2,\ 1^4$ are in a block.

$\begin{pmatrix} * & * \\ * & * \end{pmatrix}$ has no 4-hooks.

$G = GL(n,q)$, B upper triangular matrices, E the G-module induced from the trivial character of B.

The endomorphism algebra H_n of E is a Hecke algebra of type A over $\mathbf{C}$ with generators $\{T_1, T_2, \ldots T_{n-1}\}$ and some relations, e.g. $T_i^2 = (q-1)T_i + qT_i$.

For certain values of q, H_n is not semisimple and we can talk of its modular representations, decomposition numbers, etc.

Work done on blocks and decomposition matrices by methods described above: Dipper-James, Geck, Gruber, Hiss, Kessar ...

Problems: If $G = S_n$, p a prime, describe the p-modular decomposition matrix of G.

If $G = GL(n, q)$, ℓ a prime not dividing q, describe the ℓ-modular decomposition matrix of G.

New modular representation theory connects decomposition numbers for symmetric groups, Hecke algebras, with Lie theory.

Idea of "Categorification":

Replace the action of a group or algebra on a vector space by the action of functors on the Grothendieck group of a suitable abelian category.

For example, the sum of Grothendieck groups $\oplus_{n\geqslant 0} K_0(mod - kS_n)$, or $\oplus_{n\geqslant 0} K_0(mod - H_n)$, basis indexed by partitions of n.

The quantized Kac-Moody algebra $\mathcal{U}_q(\widehat{sl_e})$ over $\mathbf{Q}(q)$ is generated by $e_i, f_i, k_i, k_i^{-1}, \ldots, (0 \leqslant i \leqslant e-1)$ with some relations.

Consider the Fock space $\oplus_{n \geqslant 0} K_0(mod - FH_n)$, ($H_n$ the Hecke algebra as before), F a field of characteristic 0. Then $\mathcal{U}_q(\widehat{sl_e})$ acts on this space!

e_i, f_i are functors on the Fock space, called i-induction, i-restriction.

Work of Ariki, Grojnowski, Vazirani, Lascoux-Leclerc-Thibon, Varagnolo-Vasserot, ...

Decomposition matrix D for H_n with q a e-th root of unity, appears as transition between two bases of the Fock space.

Blocks appear as weight spaces for the subalgebra generated by the k_i.

Recent results (BS):

The quantized Kac-Moody algebra $\mathcal{U}_q(\widehat{gl_e})$ has generators e_i, f_i, k_i, k_i^{-1} as before for $\mathcal{U}_q(\widehat{sl_e})$, and v_k, k a positive integer. The v_k are described combinatorially by Leclerc-Thibon, in terms of horizontal ribbons.

Let $\mathcal{A}_n$ be the category of unipotent representations of $GL(n,q)$. Let

$$\mathcal{A} = (\oplus_{n \geqslant 0} K_0(\mathcal{A}_n)) \otimes_{\mathbf{Z}} \overline{\mathbf{Q}_l}(q),$$

sum of Grothendieck groups of the categories of unipotent representations of $GL(n,q)$ for all n. Then $\mathcal{U}_q(\widehat{gl_e})$ acts on this space. The operators v_k are quantized Lusztig maps on the Grothendieck group.

Theorem If $G_n = GL(n,q)$, let $L = G_n \times GL(k,q^e)$. Define maps $\mathcal{L}_k : \mathcal{A} \to \mathcal{A}$ by: $\chi_\lambda \to [R_L^{G_{n+ke}}(\chi_\lambda \times 1]$.

Then $\mathcal{L}_k$ coincides with the operator v_k specialized at $q = 1$. (More generally: $\mathcal{L}_\mu$where μ is a partition of k.)

Thus we connect $\mathcal{U}_q(\widehat{gl_e})$ (quantum gl) with finite GL. (cf Dipper-James)

An example of a decomposition matrix D for $n = 4$, $e = 4$:

$$\begin{pmatrix} 4|| & 1 & 0 & 0 & 0 \\ 31|| & 1 & 1 & 0 & 0 \\ 211|| & 0 & 1 & 1 & 0 \\ 1111|| & 0 & 0 & 1 & 1 \end{pmatrix}$$

From this matrix we can read:

- (Part of) transition between two bases of the Fock space as $\widehat{sl_e}$-module $(e = 4)$
- Decomposition numbers for H_n (also cyclotomic) over characteristic 0 $(n = 4)$
- (conjecturally) part of decomposition matrix of $GL(n, q)$, ℓ dividing $q^2 + 1$ $(n = 4, e = 4)$

Summary

- Known: Decomposition numbers for H_n (also cyclotomic) over characteristic 0
- Known: Decomposition numbers for $GL_n(q)$, ℓ large
- Not known: Decomposition numbers for S_n, $GL_n(q)$, all ℓ

An example of a decomposition matrix D for $n = 4$, $e = 4$:

$$\begin{pmatrix} 4|| & 1 & 0 & 0 & 0 \\ 31|| & q & 1 & 0 & 0 \\ 211|| & 0 & q & 1 & 0 \\ 1111|| & 0 & 0 & q & 1 \end{pmatrix}$$

Interpret this matrix as a matrix of "q-decomposition numbers".

Leads to: Graded representation theory of S_n, H_n, ...

See A.Kleshchev, Bulletin of AMS 47 (2010), 419-481.

Summary

- Groups of Lie type: reps constructed by induction, H-C induction, D- L induction
- Look for: Blocks, Decomposition Numbers in Modular Rep Theory of S_n, $GL(n, q)$, (cyclotomic) Hecke algebras
- Now linked with affine Kac-Moody algebras in type A.
- Leads to: Graded Representation Theory of symmetric groups, ...

End of story? We know there is no end.