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MODULAR REPRESENTATIONS, OLD AND NEW

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To the memory of Harish-Chandra

1. Introduction

The art of story telling is very old in India, as is evidenced by the great epics Ramayana and Mahabharata. Even now lectures by scholars who tell contemporary versions of these epics are very popular, as was a TV series on the Ramayana, where it is said that a train stopped at a station long enough for the passengers to get out and see the latest episode.

In this paper I would like to tell a story of modular representations of symmetric groups, Hecke algebras and related objects, starting from Brauer's introduction of the concept and describing some recent developments connecting this theory with Lie theory.

We begin with stating some conjectures in the representation theory of finite groups, some of them being long-standing, and discuss recent progress in them. We then discuss the ordinary, or characteristic 0, representation theory of finite reductive groups, including Harish-Chandra theory and Deligne-Lusztig theory, which also play a role in the modular representation theory. The main theorem in the classification of blocks in the ℓ-modular representation theory is stated.

In recent years there has been a new direction in the modular representation theory of symmetric groups and Hecke algebras via connections with Lie theory, leading to the concept of graded representation theory. A detailed, definitive account of these developments has been given by A.Kleshchev [34]. In the second part of the paper we give an introduction to these ideas, and hope that the reader will then be encouraged to read [34] and other papers in the literature.

Notation: Given an algebra A, $A-mod$ (resp. $A-pmod$) is the category of finitely generated A-modules (resp. projective A-modules). For a category $\mathcal{A}$, $K_0(\mathcal{A})$ is the Grothiendieck group. For an algebra A, $K_0(A)$ is the Grothiendieck group of the category $A - mod$.

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For a finite group G, $\mathrm{Irr}(G)$ is the set of ordinary irreducible characters of G. The usual inner product on the space of class functions on a finite group will be denoted by $\langle -, - \rangle$.

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2. Finite groups

References for this section are ([16], Ch.12). All groups in this section will be finite.

Let G be a finite group. The theory of group characters goes back to the work of Frobenius. If $\rho : G \to GL(n, K)$ is a representation of degree n over a field K, the character χ of ρ is the class function on G defined by $\chi(g) = \mathrm{Tr}(\rho(g))$. If $K = \mathbf{C}$ (or a sufficiently large field of characteristic 0) the (ordinary) irreducible characters form an orthonormal basis of the space of K-valued class functions on G. The number of ordinary characters is the number of conjugacy classes of the group, and this leads to a "character table" for the group which yields a lot of information about the group. We can now state

Problem 1.

(1) Classify and describe the (ordinary) irreducible characters of G over K as above, e.g. give their degrees.

(2) Describe the character table of G.

For some well-known groups such as the symmetric group S_n or the finite general linear group $GL(n, q)$ we can answer these questions. In a fundamental paper in 1955 J.A.Green constructed the characters of the groups $GL(n, q)$ (see e.g. T.A.Springer, Characters of special groups, in [8]).

Frobenius computed the first character table, that of $PSL(2, p)$. He also defined the important notion of induced characters, which we still use today even though there are generalizations for groups of Lie type, for example.

The theory of modular representations, i.e. the representations of G over a field k of characteristic p where p divides the order of G was developed by Richard Brauer starting in the 1930's. In this case Brauer defined characters of the irreducible representations as complex-valued functions on the p-regular classes of the group, and they are now called Brauer characters. Brauer then divided the ordinary characters into subsets called "blocks" as follows.

Let K be a sufficiently large field of characteristic 0, $\mathcal{O}$ a complete discrete valuation ring with quotient field K, and k a residue field of $\mathcal{O}$ such that the characteristic of k is p . Consider the algebras KG, $\mathcal{O}G$, kG. We have

$\mathcal{O}G = B_1 \oplus B_2 \oplus \ldots \oplus B_n$ where the B_i are "block algebras", indecomposable ideals of $\mathcal{O}G$. We have a corresponding decomposition of kG. The principal block is the one which contains the trivial character of G.

An invariant of a block B of G is the defect group, a p- subgroup P of G, unique up to G-conjugacy. One definition of P is that P is minimal with respect to: Every B-module is a direct summand of an induced module from P. The "Brauer correspondence" then gives a bijection between blocks of G of defect group P and blocks of $N_G(P)$ of defect group P.

An ordinary representation of G, i.e. a representation over K, is equivalent to a representation over $\mathcal{O}$, and can then be reduced mod p to get a modular representation of G over k. Then one can define the "decomposition matrix", the transition matrix between ordinary characters and Brauer characters. This leads to:

- a partition of the ordinary characters, or KG-modules, into blocks
- a partition of the Brauer characters, or kG-modules, into blocks
- a partition of the decomposition matrix into blocks

Example: A_5, $p = 2$: Ordinary characters in the principal block:

order of element	1	2	3	3	5
class size	1	15	20	12	12
χ_1	1	1	1	1	1
χ_2	5	1	-1	0	0
χ_3	3	-1	0	$\frac{1-\sqrt{5}}{2} - 1$	$\frac{1+\sqrt{5}}{2} - 1$
χ_4	3	-1	0	$\frac{1+\sqrt{5}}{2} - 1$	$\frac{1-\sqrt{5}}{2} - 1$

Example: A_5, $p = 2$: Brauer characters in the principal block:

order of element	1	3	3	5
class size	1	20	12	12
ψ_1	1	1	1	1
ψ_2	2	-1	$\frac{1+\sqrt{5}}{2} - 1$	$\frac{1-\sqrt{5}}{2} - 1$
ψ_3	2	-1	$\frac{1-\sqrt{5}}{2} - 1$	$\frac{1+\sqrt{5}}{2} - 1$

Decomposition matrix for Principal Block of A_5: $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

Some of the main problems in the modular theory are:

Problem 2.

(1) Describe the Brauer characters of G over a (sufficiently large) field k of characteristic p as above, e.g. give their degrees.

(2) Describe the blocks
(3) Find the decomposition matrix D
(4) Global to local: Describe information on the block B by "local information", i.e. from blocks of subgroups of the form $N_G(P)$, P a p-group

Problem 2(1) is hard, and still open even for S_n and $GL(n, q)$. We will say more about (2) and (3) later. We discuss (4) below.

The Sylow theorems are among the first facts we learn about finite groups. One of Brauer's insights was to realize the importance of what we now call local subgroups of a group G, i.e. subgroups of the form $N_G(P)$ where P is a p-subgroup of G, for p- modular representation theory. His idea is that "global information" about a p-block of G must be obtainable from "local information", i.e. the characters of $N_G(P)$ where P is the defect group of the block. A striking example of this is when the defect group of the block is cyclic (see [1], p.126).

There have been several conjectures put forward in modular representation theory relating the characters in a block of G with the characters in blocks of local subgroups of G. Of these, the simplest to state is the McKay conjecture, which asserts that for every finite group G and every prime p, the number of irreducible characters of G having degree prime to p is equal to the number of such characters of the normalizer of a Sylow p-subgroup of G. There also exists a block-wise version, the Alperin-Mckay conjecture, comparing characters in blocks of G and in the normalizer of their defect groups. The Alperin weight conjecture counts the number of Brauer characters ([14], p.96; [44]). There are various further deep conjectures in the representation theory of finite groups, the most elaborate ones being the Isaacs-Navarro conjectures, Dade's conjectures and Broué's conjectures [2], [44].

The McKay-conjecture was proved for solvable groups and more generally for p-solvable groups as well as for various classes of non-solvable groups, like the symmetric groups and the general linear groups over finite fields, but it remained open in general. Recently, Isaacs, Malle and Navarro [24] reduced the McKay conjecture to a question about simple groups and gave a list of conditions that they hoped all simple groups will satisfy. They showed that the McKay conjecture will hold for a group G if every simple group involved in G satisfied these conditions. For more on this problem see [41] and [42]. These are conjectures at the level of characters.

Let $\mathcal{O}$ be as above. Given a block B of a group G and a block b of a group H (e.g. $H = N_G(D)$, D defect group of B), a perfect isometry is a bijection between $K_0(B)$ and $K_0(b)$ preserving certain invariants of B and b, and an isotypy is a collection of compatible perfect isometries [10]. Here B and b are regarded as $\mathcal{O}$-algebras. At the level of characters, Broué's conjecture is that there is a perfect isometry between $K_0(B)$ and $K_0(b)$ where $H = N_G(D)$. At the module level we have Broué's abelian defect group conjecture [10], which we state below.

If A is an $\mathcal{O}$-algebra, $\mathcal{D}^b(A)$ is the bounded derived category of the category $A - mod$. It is a triangulated category.

Broué's Abelian Defect Group Conjecture:

Conjecture 1. *Let B be a block of G with the abelian defect group D, b the Brauer correspondent of B in $N_G(D)$. Then $\mathcal{D}^b(B)$ and $\mathcal{D}^b(b)$ are equivalent as triangulated categories.*

A discussion of the Abelian Defect Group Conjecture is in the 1998 ICM address of J.Rickard [46]. A.Marcus [44] gives the current status of the conjecture and describes some of the methods used to prove it.

3. Finite reductive groups

References for this section are [11], [12], [14], [21], [17]

Until 1955 the known finite simple groups which included the classical groups and the alternating groups were all studied separately. A fundamental paper of Chevalley [13] changed this by giving a unified treatment of finite simple groups, arising from simple Lie algebras. A treatment of this theory can be found in [12].

The modern view is as follows. Let $\mathbf{G}$ be a connected reductive group defined over $\mathbf{F}_q$, F a Frobenius endomorphism, $F : \mathbf{G} \to \mathbf{G}$. Then $G = \mathbf{G}^F$, the group of F-fixed points of F, is called a finite reductive group (or finite group of Lie type). Using the structure of reductive algebraic groups we get subgroups of $\mathbf{G}$ as follows. A torus is a closed subgroup $\mathbf{T} \simeq \mathbf{F}^\times \times \mathbf{F}^\times \times \cdots \times \mathbf{F}^\times$, where $\mathbf{F}$ is an algebraic closure of $\mathbf{F}_q$. A Levi subgroup $\mathbf{L}$ is the centralizer $\mathbf{C_G}(\mathbf{T})$ of a torus $\mathbf{T}$. A Borel subgroup $\mathbf{B}$ is a maximal connected solvable subgroup. By taking F-fixed points we get tori T, Borel subgroups B and Levi subgroups L in G. In G there may be several conjugacy classes of maximal tori, but there is, up to conjugacy, a distinguished pair $T \subset B$ of a maximal torus and a Borel subgroup in G. A subgroup containing a Borel subgroup is a parabolic subgroup of G. A parabolic subgroup P has a Levi decomposition $P = LV$ where L is a Levi subgroup and V is the unipotent radical of P.

Examples. $G = GL(n,q), U(n,q), Sp(2n,q)$, or $SO^{\pm}(2n,q)$. In $GL(n,q)$ Levi subgroups L are isomorphic to $\prod_i GL(m_i, q^{e_i})$.

The modular representation theory of finite reductive groups over a field of characteristic p where p divides q (the "defining charactistic") was in fact developed earlier than the ordinary representation theory. The work of C.Curtis on this in the setting of "BN-pairs" is described in [8]. This theory is modeled on the highest weight theory for representations of semisimple Lie algebras and Lie groups. Later work in this theory, again viewing the finite groups as coming from algebraic groups via a Frobenius morphism, is due to Anderson, Humphreys, Jantzen, Lusztig, Soergel and others. The reader is referred to [23] for this rich theory which we will not describe in this paper.

In the ordinary representation theory of finite reductive groups, Curtis, Iwahori and Kilmoyer constructed the "principal series" for G by inducing characters of a Borel subgroup and showing that the endomorphism algebra of the induced representation is a Hecke algebra (these days sometimes called Iwahori-Hecke algebras) (see [17], Section 67). This then led to Harish-Chandra theory, described below.

Let $\mathbf{P}$ be an F-stable parabolic subgroup of $\mathbf{G}$ and $\mathbf{L}$ an F-stable Levi subgroup of $\mathbf{P}$ so that $L \leq P \leq G$. Then Harish-Chandra induction is the following map: $R_L^G : K_0(KL) \to K_0(KG)$, where K is a sufficiently large field of characteristic 0 as before, such that if $\psi \in \mathrm{Irr}(L)$ then $R_L^G(\psi) = \mathrm{Ind}_P^G(\tilde{\psi})$ where $\tilde{\psi}$ is the character of P obtained by inflating ψ to P, using the Levi decomposition $P = LV$ of P.

We say $\chi \in \mathrm{Irr}(G)$ is cuspidal if $\langle \chi, R_L^G(\psi)\rangle = 0$ for any $L \leq P < G$ where P is a proper parabolic subgroup of G. The pair (L, θ) is a cuspidal pair if $\theta \in \mathrm{Irr}(L)$ is cuspidal. We then have the main theorem of Harish-Chandra induction:

Theorem 1. *(i) Let (L, θ), (L', θ') be cuspidal pairs. Then $\langle R_L^G(\theta), R_{L'}^G(\theta')\rangle = 0$ unless the pairs (L, θ), (L', θ') are G-conjugate.*

(ii) If χ is a character of G, then $\langle \chi, R_L^G(\theta)\rangle \neq 0$ for a cuspidal pair (L, θ) which is unique up to G-conjugacy. Thus Irr(G) partitioned into Harish-Chandra families: A family is the set of constituents of $R_L^G(\theta)$ where (L, θ) is cuspidal.

The endomorphism algebras of Harish-Chandra-induced cuspidal representations from parabolic subgroups were then investigated and described by Howlett and Lehrer (see [11], Ch 10). However, not all irreducible representations, in particular the cuspidal ones, were obtained this way. There were some character tables as well as the work of J.A.Green on the characters of $GL(n, q)$ which led to the idea (attributed to I.G.Macdonald by T.A.Springer in [Cusp forms in finite groups, [8]]), that there should be families of characters of G corresponding to characters of maximal tori. This was in fact what was proved in the spectacular paper of Deligne and Lusztig in 1976 using ℓ-adic cohomlogy, where ℓ is a prime not dividing q. They introduced a map $R_{\mathbf{T}}^{\mathbf{G}} : K_0(\overline{\mathbf{Q}}_l T) \to K_0(\overline{\mathbf{Q}}_l G)$ where $\mathbf{T}$ is a maximal torus of $\mathbf{G}$. Lusztig later generalized this map, replacing maximal tori by Levi subgroups, and we will describe this below.

Suppose $\mathbf{L}$ is an F-stable Levi subgroup, not necessarily in an F-stable parabolic $\mathbf{P}$ of $\mathbf{G}$. Let ℓ be a prime not dividing q. The Lusztig linear operator is a map $R_{\mathbf{L}}^{\mathbf{G}} : K_0(\overline{\mathbf{Q}}_l L) \to K_0(\overline{\mathbf{Q}}_l G)$, which has the property that every χ in $\mathrm{Irr}(G)$ is in $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ for some $(\mathbf{T}, \theta)$, where $\mathbf{T}$ is an F-stable maximal torus and $\theta \in \mathrm{Irr}(T)$.

The unipotent characters of G are the irreducible characters χ in $R_{\mathbf{T}}^{\mathbf{G}}(1)$ as $\mathbf{T}$ runs over F-stable maximal tori of $\mathbf{G}$. Here 1 is the trivial character of T. If $L \leq P \leq G$, where $\mathbf{P}$ is a F-stable parabolic subgroup, $R_{\mathbf{L}}^{\mathbf{G}}$ is just Harish-Chandra induction.

To see how the map $R_{\mathbf{L}}^{\mathbf{G}}$ is constructed using an algebraic variety on which G and L act, see [21], Ch 11.

In a series of papers and in his book [39] Lusztig classified all the irreducible characters of G, provided $\mathbf{G}$ has a connected center. (This restriction was removed by him later.) This classification leads to two new orthonormal bases of the space $\mathcal{C}(G)$ of $\overline{\mathbf{Q}}_l$-valued class functions of G: the basis of "almost characters" and the basis of characteristic functions of F-stable character sheaves on $\mathbf{G}$. Character sheaves are certain perverse sheaves in the bounded derived category $\mathcal{D}G$ of constructible $\overline{\mathbf{Q}}_l$-sheaves on $\mathbf{G}$. Lusztig then conjectured that the almost characters coincide with the characteristic functions of F-stable character sheaves on $\mathbf{G}$, up to a scalar multiple, if the characteristic p of $\mathbf{F_q}$ is "almost good". This conjecture has now been proved by T.Shoji and others in many cases, including groups $\mathbf{G}$ with a connected center (see [48]). If the conjecture is true, including the precise values of certain scalars, the character table of G is determined in principle. This shows the power of geometrical methods in representation theory. Indeed, the theory of character sheaves is an aspect of "geometric representation theory", a flourishing area of research.

4. Symmetric groups, General linear groups, Finite reductive groups

References for this section are [30],[31], [25], [14], [28], [29].

Many expositions of the ordinary representation theory of the symmetric group S_n over $\mathbf{Q}$ are available (see e.g. [30], [31]). If $\chi \in \mathrm{Irr}(G)$ then we can write $\chi = \chi_\lambda$ where λ is a partition of n. Then there is a Young diagram corresponding to λ and p-hooks, p-cores are defined, so that we can talk of the p-core of a partition λ for a prime p.

In the modular representation theory, the p-blocks were classified in the famous

Theorem 2. *(Brauer-Nakayama) ([31], p.245). The characters χ_λ, χ_μ of S_n are in the same p-block (p prime) if and only if λ and μ have the same p-core.*

However, the p-modular decomposition numbers are not known for S_n. The modular theory has taken surprising new directions which will be described later.

We now look at the ℓ-modular theory for $GL(n, q)$ where ℓ is a prime not dividing q (the "non-defining charactistic") which was started in [25] and led to the development of the theory by several authors for finite reductive groups.

Let q be odd. The ℓ-blocks were classified in [25] for ℓ odd and extended to $\ell = 2$ by M.Broué. Let e be the order of q mod ℓ. The unipotent characters of $GL(n, q)$ are parametrized by partitions of n, and we denote the character corresponding to a partition λ by χ_λ.

Theorem 3. *(Fong-Srinivasan) χ_λ, χ_μ are in the same ℓ-block of $GL(n, q)$ if and only if λ, μ have the same e-core.*

Example: $n = 5$, ℓ divides $q + 1$, $e = 2$. Then χ_λ for $5,\ 32,\ 31^2,\ 2^21,\ 1^5$ are in a block. Same for S_5, $p = 2$.

Example: $n = 4$, ℓ divides $q^2 + 1$, $e = 4$. Then χ_λ for $4,\ 31,\ 21^2,\ 1^4$ are in a block.

$\begin{pmatrix} * & * \\ * & * \end{pmatrix}$ has no 4-hooks.

The ℓ-blocks were then classified for various special cases, and finally Cabanes and Enguehard ([14], 22.9) proved the following theorem for unipotent blocks, which are in some sense building blocks for all ℓ-blocks.

Definition. A unipotent block of G is a block which contains unipotent characters.

Theorem 4. *(Cabanes-Enguehard) Let B be a unipotent ℓ-block of G, ℓ odd. Then the characters in B are a union of Lusztig families, i.e. a union of constituents of $R_L^G(\psi)$ for various pairs (L, ψ).*

This theorem leads to the following

Surprise: Brauer Theory and Lusztig Theory are compatible!

We have now described a satisfactory solution to Problem 2 (2), i.e. the classification of ℓ-blocks for finite reductive groups. We now consider Problem 2 (3), the decomposition matrix. In the case of $GL(n, q)$ the decomposition matrix is related to that of a q-Schur algebra, and this will be described in Section 7. In the case of classical groups, work has been done by Geck, Gruber, Malle and Hiss where ℓ is a so-called "linear prime", using a modular version of Harish-Chandra theory [28]. The problem of finding the decomposition matrix in general is still open, even for unitary groups. For more recent work here see [29].

5. Weyl groups, Cyclotomic Hecke algebras, q-Schur algebras

References are [3], [20], [43].

Our story continues with the introduction of some more major players.

Weyl groups play an important role in the theory of finite reductive groups. For example, the conjugacy classes of maximal tori in a finite reductive group G are parametrized by the F-conjugacy classes in the Weyl group (see [11], 3.3). There is a set of class functions on G known as "almost characters" which involves the characters of W and plays an important role in the character theory of G, as mentioned in Section 3. The ordinary characters of Weyl groups are understood, but as mentioned above the modular theory has open problems even for the type A case, i.e. S_n.

The Hecke algebras which are deformations of Weyl groups and which arise as endomorphism algebras of induced representations of finite reductive groups were studied by Curtis, Iwahori and Kilmoyer ([17], Ch 8). These Hecke algebras and their characters are studied from the point of view of symmetric algebras in the book by M.Geck and G.Pffeifer [26]. With each Hecke algebra we have certain polynomials called generic degrees, which when specialized give the degrees of the constituents of the induced representations mentioned above. G.Lusztig also showed that more complicated Hecke algebras occur as endomorphism algebras of Harish-Chandra induced cuspidal representations of Levi subgroups, and used these results to classify the unipotent representations. Variants of these Hecke algebras also occur in the work of Howlett and Lehrer on more general induced representations ([11], Ch 10).

Later Ariki-Koike and Broué-Malle introduced cyclotomic Hecke algebras which are deformations of complex reflection groups [5]. These algebras also arise in the Deligne-Lusztig theory in a mysterious way, that is, they behave as though their specializations are endomorphism algebras of the virtual representations $R_{\mathbf{L}}^{\mathbf{G}}$ defined above. For example, it has been observed that there is a bijection with signs between the constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ for a suitable pair (L, λ) and characters of a complex reflection group ([9], 3.2). In the paper [9] there is developed a theory of "generic groups" which are defined with respect to root data, and the Weyl group plays a central role.

The cyclotomic Hecke algebras which we are interested in are deformations of the groups denoted $G(m, 1, n)$ by Ariki. The group $G(m, 1, n)$ is isomorphic to $\mathbf{Z}_m \wr S_n$.

Definition ([34], p.420; [3], 12.1). The cyclotomic Hecke algebra $\mathcal{H}_n$ over a field F has generators $T_1, T_2, \ldots T_n$ and parameters $q, v_1, v_2, \ldots v_m$ which satisfy the relations ([3], 12.1):

$(T_1 - v_1)(T_1 - v_2) \ldots (T_1 - v_m) = 0, (T_i - q)(T_i + q) = 0 \ (i \geq 2),$

$T_1T_2T_1T_2 = T_2T_1T_2T_1, T_iT_j = T_jT_i$ $(i \geq j+2)$,

$T_iT_{i-1}T_i = T_{i-1}T_iT_{i-1}$ $(3 \leq i \leq n)$.

In particular we get the Hecke algebra of type B or C if $m = 2$. It is a classical result that the ordinary representations of $G(m, 1, n)$ are parametrized by m-tuples of partitions whose sizes add up to n. The analogous result holds for the ordinary representations of $\mathcal{H}_n$ in the semisimple case. Now $\mathcal{H}_n$ is semisimple if and only if $q^i v_j - v_k$ $(|i| < n,\ j \neq k)$ and $1+q+\ldots q^i$ $(1 \leq i < n)$ are all non-zero (see [5], 2.9). In the non-semisimple case of interest to us where F has characteristic 0, the v_i are powers of q and q is an e-th root of unity, we have to take e-restricted partitions (also called Kleshchev partitions) when $m = 1$, and Kleshchev multi-partitions when $m > 1$, to parameterize the irreducible modules (see [3], 12.1, 12.2). Here a partition λ is e-restricted if $\lambda_i - \lambda_{i+1} < e$ for all $i \geq 1$, where the λ_i are the parts of λ.

In order to study the decomposition numbers for $GL(n, q)$ Dipper and James [20] introduced the q-Schur algebra. In the classical Schur-Weyl theory the Lie group $GL(n, \mathbf{C})$ and S_d act as centralizers of each other on a tensor space. The Schur algebra $S_{d,n}$ which has a basis indexed by partitions of n with no more than d parts is the image of the action of $GL(n, \mathbf{C})$ on the tensor space. The q-Schur algebra is a deformation of the Schur algebra, defined by Dipper and James as the endomorphism algebra of a sum of permutation modules for the Hecke algebra H_n of type A corresponding to S_n.

The q-Schur algebra $\mathcal{S}_q(n)$ is defined over any field F and involves an element $q \in F$. We use the same q for H_n.

Definition ([43], p.55). $\mathcal{S}_q(n) = \mathrm{End}_{H_n} \oplus_\mu M^\mu$.

Here μ runs over the partitions of n, and M^μ is a permutation module for H_n.

At this stage we mention another important object, the affine Hecke algebra. This has been studied for a long time, but is of special interest here. In a paper of Ram and Ramagge [47] it is shown how the representation theory (in characteristic 0) of Hecke algebras of classical type can be derived from the representation theory of the affine Hecke algebra of type A. Indeed, there is also a flourishing "Combinatorial Representation Theory", see [7].

6. Lie algebras

References are [22], [32] and [33].

In the classical Cartan-Killing theory, finite-dimensional semisimple Lie algebras are classified by Cartan matrices and Dynkin diagrams. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. There is a root lattice and a weight lattice associated with $\mathfrak{g}$. The root lattice has a basis of simple roots, and there is a concept of dominant weights with respect to this basis. The weight lattice has a basis of fundamental dominant weights, in duality with the basis of simple roots. The Cartan subalgebra $\mathfrak{h}$ acts on a finite-dimensional irreducible module of $\mathfrak{g}$ by linear functions, also called weights, of which one is a "highest weight". The module is then a direct sum of weight spaces

V.Kac and B.Moody generalized this theory to construct infinite-dimensional semisimple Lie algebras which are now known as Kac-Moody algebras. In this theory a generalized Cartan matrix A is a real matrix of the form $A = (a_{ij})$ where $a_{ii} = 2$, a_{ij} are non-positive integers if $i \neq j$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$. Then A is said to be symmetrizable if $A = DB$ where D is diagonal and B is symmetric. The classical representation theory was also extended to the Kac-Moody case, but now we have to consider infinite-dimensional modules. In order to get a theory resembling the finite-dimensional case one considers modules in the category $\mathcal{O}$. Then again the module is a direct sum of weight spaces for the Cartan subalgebra (see [33], Ch 9).

Given a generalized Cartan matrix A there are (i) a semisimple Lie algebra (Kac-Moody algebra) $\mathfrak{g} = \mathfrak{g}(A)$, (ii) a universal enveloping algebra $U(\mathfrak{g})$, and (iii) a quantum enveloping algebra (a Hopf algebra) $U_q(\mathfrak{g})$. The universal enveloping algebra of a finite-dimensional semisimple Lie algebra is a classical object, but the quantum enveloping algebra $U_q(\mathfrak{g})$ has been studied since the 1980's and is now a part of Lie theory. Its representation theory mirrors the classical theory.

Given a Kac-Moody algebra $\mathfrak{g}$ we have a weight lattice P with a basis $\{\Lambda_i,\, i \in I\}$ of fundamental dominant weights and the notion of a dominant integral weight in P. If Λ is a dominant integral weight then $\mathfrak{g}$ has an irreducible integrable module $V(\Lambda)$ which is the direct sum of weight spaces for the Cartan subalgebra ([33], Chs 9 and 10).

The quantum enveloping algebra $U_q(\mathfrak{g})$ is generated by $E_i, F_i, K_i^{\pm 1}$, $i \in I$, with some relations. We have a triangular decomposition (isomorphism of vector spaces) $U_q(\mathfrak{g}) = {U_q}^+(\mathfrak{g}) \otimes {U_q}^0(\mathfrak{g}) \otimes {U_q}^-(\mathfrak{g})$ (see [32], 4.3 and Theorem 4.21, [3], p.21).

In the case of the quantum enveloping algebra $U_q(\mathfrak{g})$, for Λ as above we have again an integrable module $V(\Lambda)$ which is the direct sum of weight spaces for ${U_q}^0$, the subalgebra generated by the $K_i^{\pm 1}$. Then $V(\Lambda)$ is irreducible if q is not a root of unity.

G. Lusztig proved a remarkable result regarding represenations of quantum enveloping algebras and thus of reductive Lie algebras . He defined a "canonical basis" for such an algebra $\mathfrak{g}$ with some remarkable positivity properties, which gives rise to a distinguished basis again called a canonical basis of any irreducible $\mathfrak{g}$-module (see [3], 7.1). Kashiwara has a similar basis known as the crystal basis.

7. Modular Representations, New

References are [43], [20]

New modular representation theory connects decomposition numbers for symmetric groups, cyclotomic Hecke algebras and q-Schur algebras with Lie theory.

We will first discuss what is meant by "modular representations" for the various objects we have introduced: (1) Finite groups of Lie type (2) Hecke algebras (3) q-Schur algebras.

For (1) the method of reduction mod p has already been described, and this can be applied to S_n and $GL(n, q)$. In the case of S_n we have the classical Specht modules which can be defined over any field.

(2) As in ([43], p.133), ([3], 12.1) consider $\mathcal{H}_n$ over a field F. We have elements $q \neq 0$ and $v_1, v_2, \ldots v_m$ in F, and a presentation for $\mathcal{H}_n$. Then a "Specht module" can be defined for $\mathcal{H}_n$ corresponding to a multipartition, which is irreducible over F in the semisimple case (see Section 5). In the non-semisimple case the Specht module is no longer irreducible and we can look at its composition factors and talk of the decomposition matrix. When F has characteristic 0 and q is a root of unity Ariki's Theorem, which will be discussed in Section 8, describes the decomposition matrices.

(3) As in (2) a"Weyl module" can be defined for $\mathcal{S}_q(n)$, and its composition factors when q is an e-th root of unity give rise to a decomposition matrix. Then Dipper and James proved a remarkable relationship between the decomposition matrices of $\mathcal{S}_q(n)$, $GL(n, q)$ and S_n ([43], 6.47). Their theorem states that the decomposition matrix of $\mathcal{S}_q(n)$, now defined over a field F of characteristic ℓ, with e the order of q mod ℓ, is a submatrix of the decomposition matrix of $FGL(n, q)$. In fact, it is the same as the part of the decomposition matrix of $FGL(n, q)$ corresponding to the unipotent representations. An interesting fact is that both these matrices are square matrices; in the case of $GL(n, q)$ this fact is known by [25].

However, by the work of Varagnolo-Vasserot ([49],[43]), we know the decomposition matrix of $\mathcal{S}_q(n)$ only in characteristic 0. One then has the James Conjecture, which is still open, which relates the decomposition matrices in characteristic 0 and characteristic ℓ by means of an "adjustment matrix" ([43], p.115).

Example, Source: GAP An example of a decomposition matrix D for $\mathcal{S}_q(n)$, $n = 4$, $e = 4$: $\begin{pmatrix} 4|| & 1 & 0 & 0 & 0 \\ 31|| & 1 & 1 & 0 & 0 \\ 211|| & 0 & 1 & 1 & 0 \\ 1111|| & 0 & 0 & 1 & 1 \end{pmatrix}$

8. Introducing Lie Theory

References are [3], [43], [27].

The story now continues with a classic paper of Lascoux, Leclerc and Thibon on the Hecke algebra H_n of type A, defined over a field of characteristic 0. They gave an algorithm to compute a matrix, conjectured to be the decomposition matrix of H_n when the parameter q is an e-th root of unity, and connected it with the affine Kac-Moody algebra $\widehat{sl_e}$ (see [43], p.95). Ariki proved the conjecture, in fact for cyclotomic Hecke algebras. We state his theorem as in ([3], Theorem 12.5).

Consider the cyclotomic Hecke algebra $\mathcal{H}_n$ over a field F of characteristic 0 associated with the group $G(m, 1, n)$ with parameters $q, v_1, v_2, \dots v_m$, where each $v_i = q^{\gamma_i}$, $\gamma_i \in \mathbf{Z}/\mathbf{eZ}, i = 1, 2, \dots m$ and q is an e-th root of unity, not equal to 1. Let $\Lambda = \sum_i n_i\Lambda_i$, where v_i occurs with multiplicity n_i. Let $V(F)$ be the complexified Grothiendieck group $\oplus_{n\geq 0}K_0(\mathcal{H}_n - \text{pmod}) \otimes_{\mathbf{Z}} \mathbf{C}$.

Remark. Here all the $\mathcal{H}_n$ are assumed to have the same parameters. Note that q is a special parameter. Then

Theorem 5. *(Ariki). The Kac-Moody Lie algebra $\widehat{sl_e}$ acts on $V(F)$ and $V(F)$ is isomorphic to the irreducible $\widehat{sl_e}$-module of highest weight Λ. The canonical basis (in the sense of Lusztig) of the module, specialized at $q = 1$ corresponds under the isomorphism to the basis of $V(F)$ of indecomposable projective modules of the $\mathcal{H}_n$.*

Thus, we introduce Lie theory into the modular representation theory of the Hecke algebras $\mathcal{H}_n$ taken over all n. Now we also have a standard basis for $V(F)$, corresponding to the irreducible modules of $\mathcal{H}_n$. We then have the important fact that the transition matrix between the two bases gives the decomposition matrices for the e-modular theory of the $\mathcal{H}_n$. Furthermore, the decomposition of $V(F)$ into weight spaces (of the Cartan subalgebra) of $\widehat{sl_e}$ corresponds to the decomposition into blocks for the $\mathcal{H}_n$, which is another amazing fact.

The algebra $\widehat{sl_e}$ has generators e_i, f_i, $i = 1, 2, \dots e$. Their action on $V(F)$ is given by functors called i-restriction, i-induction which can be described combinatorially.

It is worth noting that in the case of the symmetric groups, these operations were defined by G.de B. Robinson, a pioneer in the representation theory of S_n (see [31], p.271). In that case, i-restriction, i-induction correspond to ordinary restriction and induction followed by cutting to a block. We describe these operations below.

Irreducible KS_n-modules are indexed by partitions of n, and a partition of n is represented by a diagram with n nodes. Induction (resp. restriction) from S_n to S_{n+1} (resp. S_n *to* S_{n-1}) is combinatorially represented by adding (resp. removing) a node from a partition.

Now fix an integer $\ell \geq 2$. The residue r at the (i, j)-node of a diagram is defined as $r \equiv (j - i)(mod\ \ell)$. We define induction (resp. restriction) from S_n to S_{n+1} (resp. S_n *to* S_{n-1}) by defining operators e_i and f_i, $0 \leq i \leq (\ell - 1)$, which move only nodes with residue i. Then Ind $= \sum_0^{l-1} f_i$ and Res $= \sum_0^{l-1} e_i$. We have similar operations, with multipartitions, in the case of $\mathcal{H}_n$ (see [27] for a discussion).

We now see that given the Hecke algebra $\mathcal{H}_n$ with certain parameters, we have a dominant weight Λ for $\widehat{sl_e}$ associated with it. Thus the Hecke algebra can be parametrized by Λ. In Ariki's theorem we may denote the Hecke algebra as $\mathcal{H}_n^\Lambda$ and the module $V(F)$ as $V(\Lambda)$. This is the view adopted in [34].

The other main idea arising from the work of Lascoux, Leclerc and Thibon and of Ariki is the decomposition of $V(F)$ into blocks of the $\mathcal{H}_n^\Lambda$ coincides with the decomposition of the $\widehat{sl_e}$-module into weight spaces. This leads to a new notation for the blocks: blocks of $\mathcal{H}_n^\Lambda$ can be parametrized by $\nu = \sum_{i \in I} m_i i$, where the m_i are non-negative integers and $I = \mathbf{Z}/\mathbf{eZ}$ ([35], 8.1).

9. Categorification

A reference is [45]

Our story takes an abstract direction in this section. "Categorification" is indeed one of the buzzwords of the last few years. The idea is that set-theoretic notions are replaced by category-theoretic notions (see [45]). We state here a definition given in [37].

Let A be a ring, B a left A-module. Let a_i be a basis of A. One should find an abelian category $\mathcal{B}$ such that $K_0(\mathcal{B})$ is isomorphic to B, and exact endofunctors F_i on $\mathcal{B}$ which lift the action of the a_i on B, i.e. the action of $[F_i]$ on $K_0(\mathcal{B})$ descends to the action of a_i on B so that the following diagram commutes. The map $\phi : K_0(\mathcal{B}) \to B$ is an isomorphism.

$$\begin{array}{ccc} K_0(\mathcal{B}) & \xrightarrow{[F_i]} & K_0(\mathcal{B}) \\ \phi \big\downarrow & & \phi \big\downarrow \\ B & \xrightarrow{a_i} & B \end{array}$$

Example. Ariki's Theorem. Here $A = \widehat{sl_e}$, $B = V(F)$ and $\mathcal{B} = \oplus_{n \geq 0} (\mathcal{H}_n - \text{pmod})$.

We return to Broué's Abelian Defect Group Conjecture, stated in Section 2. Chuang and Rouquier proved the conjecture for S_n and $GL(n, q)$ by introducing the concept of SL_2-categorification. On the way they prove that if two blocks of symmetric groups (possibly not the same S_n) have isomorphic defect groups, the block algebras are derived equivalent, i.e. the derived categories of the module categories are equivalent [18]. This has shown the power of categorification in a seemingly unrelated area such as finite groups. For a discussion of categorical equivalences in the modular representation theory of finite groups, see [44].

10. KLR-algebras: the diagrammatic approach

References are [36], [40]

We have seen that the algebra $\widehat{sl_e}$ is related to the category $\sum_{n\geq 0}(\mathcal{H}_n -$ pmod). We have the following question: Suppose we want to replace $U(\widehat{sl_e})$ by $U(\mathfrak{g})$, where $\mathfrak{g}$ is an arbitrary semisimple Lie algebra. Then how will we define $\mathcal{B}$ to give a categorification as above?

Now we come to the papers of Khovanov-Lauda, which have connections with Knot Theory. Given a symmetrizable Cartan matrix A, there is a Kac-Moody algebra $\mathfrak{g}$ corresponding to A , the "negative part" of the quantum enveloping algebra $U_q{}^-(\mathfrak{g})$ over $\mathbf{Q}(q)$ and an integral form $\mathfrak{F}$ of $U_q{}^-(\mathfrak{g})$, a ring defined over $\mathbf{Z}[q, q^{-1}]$. As before (see Section 6) I is an index set for the fundamental weights or the simple roots. Khovanov and Lauda then construct an algebra R corresponding to this data as follows (see [40]).

A *diagram* is a collection of (planar) arcs connecting m points on a horizontal line with m points on another horizontal line. Arcs are labeled by elements of I. Let $\mathbb{N}[I] = \langle \nu | \nu = \sum \nu_i i, \nu_i \in \mathbf{N}\rangle$. Khovanov and Lauda first define rings $R(\nu)$ for each $\nu \in \mathbb{N}[I]$. The ring $R(\nu)$ is generated by diagrams in which ν_i arcs have the same label i. The product is defined by concatenation of diagrams, subject to some relations. Then $R = \oplus_\nu R(\nu)$ and $K_0(R) = \oplus_\nu K_0(R(\nu))$.

Theorem 6. *The category $R-$pmod categorifies $\mathfrak{F}$. In other words, we may take $A = \mathfrak{F}$, $B = \mathfrak{F}$, $\mathcal{B} = R -$ pmod in the commutative diagram describing the categorification. In particular, there is an isomorphism between $K_0(R)$ and $\mathfrak{F}$.*

Rouquier defined these algebras independently and they are now called Khovanov-Lauda-Rouquier (KLR) algebras, or sometimes as quiver Hecke algebras. An important property of these algebras is that they are naturally graded. This has ramifications in "graded representation theory" ([34], 2.2), to be described in the next section.

11. Graded Representation Theory

Ariki's theorem was regarding representations of $\widehat{sl_e}$, but it can be generalized to those of $U_q(\widehat{sl_e})$ ([3], 10.10). This leads to the idea of graded representation theory, where multiplication by q represents a shift in grading. Indeed, in ([34], p.431) the classification of graded irreducible modules of cyclotomic Hecke algebras is stated as the Main Problem. One can also talk of graded decomposition numbers.

We now briefly mention some of the recent important work in this theory; references for (1) and (2) are references [38], [40] and [42] in [34].

(1) (Brundan and Kleshchev) Blocks of cyclotomic Hecke algebras are isomorphic to blocks of Khovanov-Lauda-Rouquier algebras. Since the latter are graded, there is a $\mathbf{Z}$-grading of blocks of cyclotomic Hecke algebras, including group algebras of symmetric groups in positive characteristic.

(2) Building on the above, Brundan, Kleshchev and Wang have constructed gradings of Specht modules of cyclotomic Hecke algebras. Brundan and Kleshchev then describe graded decomposition numbers of these Specht modules over fields of characteristic 0.

(3) Using the connection between the Hecke algebra H_n and the q-Schur algebra $\mathcal{S}_q(n)$, Ariki [6] has shown that $\mathcal{S}_q(n)$ and its Weyl modules can be graded. Moreover, he shows that certain polynomials in v introduced by Leclerc and Thibon give the graded decomposition numbers for $\mathcal{S}_q(n)$ over a field of characteristic 0 with q an e-th root of unity. Here we consider the algebra $U_v(\widehat{sl_e})$ with a parameter v, as distinct from the parameter q in $\mathcal{S}_q(n)$.

Example. Graded decomposition matrix for $n = 4$, $e = 4$: $\begin{pmatrix} 4|| & 1 & 0 & 0 & 0 \\ 31|| & v & 1 & 0 & 0 \\ 211|| & 0 & v & 1 & 0 \\ 1111|| & 0 & 0 & v & 1 \end{pmatrix}$

Remark. If $q = 1$ we get the usual decomposition matrix given in Section 7. Other decomposition matrices are found in ([43], Appendix). The above matrix corresponds to a block, and the reader can make the connection with the theorem of [25], stated in Section 4; these are the partitions with an empty 4-core.

12. Higher Representation Theory

A reference is [45].

Let $\mathcal{C}$ be a 2-category. Then for any $i, j \in \mathcal{C}$ the morphisms form a category whose objects are called 1-morphisms and morphism are called 2-morphisms of $\mathcal{C}$.

We recall the definition of Categorification in Section 9, where the action of the algebra A on B was lifted to an action by endofunctors on a category $\mathcal{B}$. In a good situation the image of this lift would be a nice subcategory of the category of endofunctors on $\mathcal{B}$, and we can compose these endofunctors. Here we have an example of a 2-category with one object, whose 1-morphisms are functors and 2-morphisms are natural transformations (see [45], 2.4).

The concept of "Higher Representation Theory" was introduced by Chuang and Rouquier [18] and continued by Rouquier and Khovanov-Lauda. Here a 2-category is defined corresponding to a Cartan matrix and is a categorification or a "2-analogue" of the enveloping algebras $U(\mathfrak{g})$ and $U_q(\mathfrak{g})$ where $\mathfrak{g}$ is a semisimple Lie algebra. The idea then is to study "2-representations" of a 2-category.

One of the first explicit examples here was a categorification of $U_q(\widehat{sl_2})$ due to Lauda [38]. B.Webster has constructed 2-categories corresponding to $U_q(\mathfrak{g})$ where $\mathfrak{g}$ is a semisimple Lie algebra and categorifies tensor products of irreducible representations of $U_q(\mathfrak{g})$. A generalization of Ariki's theorem in this context can be found in ([50], 5.11).

13. End of Story?

We have come to the end of our story. However, as we all know there is no end to mathematical stories, and future generations will continue them.

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