Quadratic unipotent blocks of general linear, unitary and symplectic groups

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$G$ is a connected reductive algebraic group defined over $\mathbb{F}_q$, 
$F : G \rightarrow G$ a Frobenius morphism, 
$G = G^F$ is a finite reductive group.

Examples: $GL(n, q)$, $U(n, q)$, $Sp(2n, q)$, $SO^{\pm}(2n, q)$

$G$ has subgroups maximal tori, Levi subgroups (centralizers of tori)
Let $\ell$ be a prime not dividing $q$.
Suppose $L$ is an $F$-stable Levi subgroup.

- **The Deligne-Lusztig linear operator:**
  
  $R^G_L : K_0(\mathbb{Q}_L) \to K_0(\mathbb{Q}_G)$.

- The unipotent characters of $G$ are the irreducible characters $\chi$ in $R^G_T(1)$ as $T$ runs over $F$-stable maximal tori of $G$.

If $L$ is in an $F$-stable parabolic subgroup $P$,

$R^G_L$ is just Harish-Chandra induction.
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If $L$ is in an $F$-stable parabolic subgroup $P$,
$R^G_L$ is just Harish-Chandra induction.
Lusztig classification of complex characters is in good shape.
\[
\text{Irr}(G) = \bigcup \mathcal{E}(G, (s)), \text{ union of Lusztig series, } (s) \subset G^*, \text{ a semisimple conjugacy class.}
\]
$K$ a sufficiently large field of characteristic 0
$\mathcal{O}$ a complete discrete valuation ring with quotient field $K$

The ordinary characters or $KG$-modules are partitioned into blocks corresponding to the decomposition of $\mathcal{O}G$ into indecomposable two-sided ideals called block algebras.
Classification of blocks

$G$ is a finite reductive group, e.g. a classical group. $\ell$ a prime not dividing $q$.

Problem: Describe the $\ell$-blocks of $G$.

A unipotent block is a block which contains unipotent characters. Describe the unipotent blocks.
Let $G = GL(n, q)$, $e$ the order of $q$ mod $\ell$. The unipotent characters of $G$ are constituents of the permutation representation on the cosets of the subgroup $B$ of upper triangular matrices. They are indexed by partitions of $n$. Say $\chi_\lambda$ corresponds to the partition $\lambda$.

**Theorem (Fong-Srinivasan, 1982)** $\chi_\lambda, \chi_\mu$ are in the same $\ell$-block if and only if $\lambda, \mu$ have the same $e$-core.

Proof involves Deligne-Lusztig theory and Brauer theory. These two theories are compatible!
\[ G = Sp(2n, q), \ SO(2n + 1, q), \ SO^\pm(2n, q), \]

A symbol \( \Lambda \) is a pair \( (S, T) \) of subsets of \( \mathbb{N} \).

Notion of e-hooks, e-cohooks, e-cores of symbols defined.

\[
\begin{pmatrix}
0 & 1 & 2 \\
1 & 3 & \end{pmatrix},
\begin{pmatrix}
0 & 1 & 4 \\
1 & 3 & \end{pmatrix},
\begin{pmatrix}
0 & 1 & 1 \\
1 & 3 & 4 & \end{pmatrix}
\]

The second symbol comes from the first by adding a 2-hook.

The third symbol comes form the first by adding 2-cohook.
In $G = \text{CSp}(2n, q), \text{SO}(2n + 1, q), \text{CSO}^\pm(2n, q)$, unipotent characters are parameterized by symbols.

$q$ and $\ell$ odd, $e$ the order of $q$ mod $\ell$.

Unipotent blocks are again classified by $e$-cores of symbols. (Fong-Srinivasan, 1989)

**THEOREM** $\psi_{\Lambda_1}, \psi_{\Lambda_2}$ are in the same $\ell$-block if and only if the symbols $\Lambda_1, \Lambda_2$ have the same $e$-core.
e-Harish-Chandra theory for unipotent characters: The Lusztig series $\mathcal{E}(G, 1)$ is partitioned into families.

The characters in a family are constituents of $R_L^G(\psi)$ where $L$ is an “e-split Levi subgroup”, $\psi$ a unipotent “e-cuspidal” character of $L$. Then $(L, \psi)$ is called an e-cuspidal pair.

$e = 1$ gives the usual Harish-Chandra theory.
THEOREM (Cabanes-Enguehard) Let $B$ be a unipotent block of $G$, $\ell$ odd. Then the unipotent characters in $B$ are precisely the constituents of $R^G_L(\psi)$ where the pair $(L, \psi)$ is $e$-cuspidal.

Thus we have a fit of Brauer theory and Lusztig theory. The subgroup $N_G(L)$ here plays the role of a "local subgroup".

EXAMPLE. $GL(n,q)$: $L \cong T_1 \times T_2 \times \ldots T_r \times GL(m,q)$, where the $T_i$ are tori of order $q^e - 1$ and $\psi = 1 \times \chi_\lambda$, $\lambda$ an $e$-core.
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Arbitrary $\ell$-block $B$ of $G$ determines a conjugacy class $(s)$ in a dual group $G^*$ of $G$, where $s \in G^*$ is an $\ell'$-semi simple element. Then one hopes for a Jordan decomposition of blocks, i.e. a unipotent block of $C_{G^*}(s)$ sharing some properties with $B$. 
Some modern problems of modular representation theory:

$G$ is a finite reductive group, $H$ some related group, e.g. another finite reductive group, $N_G(L)$, $L$ Levi in $G$, or $C_G^*(s)$ for some $s$.

Block $B$ of $G$, block $b$ of $H$

- (Broué) Establish a perfect isometry between $B$ and $b$ (over $K$)
- (BADC) (Broué’s abelian defect group conjecture) derived equivalence of blocks between $\mathcal{O}B$ and $\mathcal{O}b$
- Morita equivalence between $\mathcal{O}B$ and $\mathcal{O}b$
Bonnafe-Rouquier: If $B$ corresponds to $s \in G^*$ where $C_{G^*}(s)$ is contained in a Levi subgroup, there is a Morita equivalence between $B$ and a unipotent block of $b$. 
Block $B$ of $G$, block $b$ of $H$:
A perfect isometry is a bijection between $K_0(B)$ and $K_0(b)$ preserving certain invariants of $B$ and $b$.
Leads to:

- $B$ and $b$ have the same number of ordinary and modular irreducible characters
- Cartan matrices of the blocks $B$ and $b$ define the same integral quadratic form.
Some results on perfect isometries, when the defect group of the blocks are abelian:

- Broué, Malle, Michel: perfect isometries between unipotent blocks of finite reductive groups and normalizers of Levi subgroups (abelian defect groups)
- Rouquier: Between two symmetric groups ("equal weight")
- Enguehard: Between two general linear groups ("equal weight")

Stronger results due to Chuang-Rouquier: BADC for general linear groups
Question: Perfect isometries between groups of different Lie types? A possibility?

Example: $GL(4, q)$ and $Sp(4, q)$, $\ell$ divides $q + 1$. There is one block correspondence between principal blocks. But $GL(4, q)$ has 5 unipotent characters in one block, $Sp(4, q)$ has 6 unipotent characters, 5 in one block and 1 in one block.
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Inspiration

$p$-adic groups, James Arthur: ”We shall describe a classification of automorphic representations of classical groups in terms of those of general linear groups (endoscopic group)”

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Waldspurger's papers
Enlarge the set of unipotent characters.

$G$ is a finite reductive group.

$Irr(G) = \bigcup \mathcal{E}(G, (s))$, union of Lusztig series, $(s) \subset G^*$, a semisimple conjugacy class.

(Waldspurger) If $s^2 = 1$, characters in $\mathcal{E}(G, (s))$ are called quadratic unipotent (special case: unipotent, $s = 1$).
$G_n = GL(n, q)$ or $U(n, q)$, $q$ odd.

Quadratic unipotent characters are parameterized by pairs of partitions $(\mu_1, \mu_2)$ of $k_i$, $i = 1, 2$ resp., with $k_1 + k_2 = n$.

$H_n = Sp(2n, q)$. Unipotent characters parameterized by (equivalence classes of) symbols.

Quadratic unipotent characters parameterized by (equivalence classes of) pairs of symbols $(\Lambda_1, \Lambda_2)$ where

$\Lambda_1$: unordered symbol of rank $k_1$

$\Lambda_2$: ordered symbol of rank $k_2$, $k_1 + k_2 = n$.

$C_{G^*}(s)$ can be disconnected, e.g $(SO(2k_1 + 1) \times SO^\pm(2k_2)) \rtimes Z_2$
Notation: $\text{Irr}(G_n)_{qu}$, $\text{Irr}(H_n)_{qu}$ for quadratic unipotent characters, $W_n$ is the Weyl group of type $B_n$.

Waldspurger’s Parametrization of $\text{Irr}(G_n)_{qu}$:

$$(\mu_1, \mu_2) \leftrightarrow \{(m_1, m_2, \rho_1, \rho_2)\}$$

$m_1, m_2 \in \mathbb{N}, \rho_i \in \text{Irr}(W_{N_i}), i = 1, 2$

$$m_1(m_1 + 1)/2 + m_2(m_2 + 1)/2 + 2N_1 + 2N_2 = n$$

Here $m_i, \rho_i$ come from the 2-core and the 2-quotient of $\mu_i$. 
Example:

$\chi_{(21,1)}$ is 2-cuspidal (no 2-core), in $\text{Irr}(GL(4))_{qu}$. In $GL(6)$, $\chi_{(41,1)}$ is obtained by Lusztig induction from $L = GL(4) \times T_{q^2-1}$, with $\rho_1 = (1, -)$. Then $(m_1, m_2, \rho_1, \rho_2) = (2, 1, (1, -), -)$.

\[
\begin{pmatrix}
* & * & * & * & *
\end{pmatrix}
\rightarrow
\begin{pmatrix}
* & * & + & +
\end{pmatrix}
\rightarrow
\begin{pmatrix}
* & *
\end{pmatrix}
\]
Waldspurger’s Parametrization of $\text{Irr}(H_n)_{qu}$:

$$\text{Irr}(H_n)_{qu} \leftrightarrow \{(h_1, h_2, \rho_1, \rho_2)\}$$

$$h_1 \in \mathbb{N}, h_2 \in \mathbb{Z}, \rho_i \in \text{Irr}(W_{N_i}), i = 1, 2$$

$$h_1(h_1 + 1) + h_2^2 + N_1 + N_2 = n$$
Waldspurger’s bijection:

\((m_1, m_2) \leftrightarrow (h_1, h_2)\), where

\[m_1 = \sup(h_1 + h_2, h_1 - h_2 - 1), \quad m_2 = \sup(h_1 - h_2, h_2 - h_1 - 1)\]

\[
\{2 \text{ – cuspidals } \in \text{Irr}(G_n)_{qu}\} \leftrightarrow \{1 \text{ – cuspidals } \in \text{Irr}(H_n)_{qu}\}\]
Extend bijection to

\[ \{\text{Irr}(G_n)_{qu}\} \leftrightarrow \{\text{Irr}(H_m)_{qu}\} \]

by

\[ \{(m_1, m_2, \rho_1, \rho_2)\} \leftrightarrow \{(h_1, h_2, \rho_1, \rho_2)\} \]

\[ m_1(m_1 + 1)/2 + m_2(m_2 + 1)/2 + 2N_1 + 2N_2 = n \]

, 

\[ h_1(h_1 + 1) + h_2^2 + N_1 + N_2 = m \]
Example: $|\text{Irr}(Sp(4, q))_{qu}| = 23$, bijection of 14 with $GL(4, q)$, 8 with $GL(3, q)$, 1 with $GL(2, q)$.

$\theta_{10} \in \text{Irr}(Sp(4, q)) \leftrightarrow \chi_{(1,1)} \in \text{Irr}(GL(2, q))_{qu}$

Note $\theta_{10}$ unipotent, $\chi_{(1,1)} \in \mathcal{E}(G, (s))$ with $s$ of order 2, $m_1 = 1, m_2 = 1, h_1 = 1, h_2 = 0$. 
Example: \(|\text{Irr}(GL(4, q))_{qu}| = 20\), bijection of 14 with \(Sp(4, q)\), 4 with \(Sp(6, q)\), 2 with \(Sp(8, q)\).

Two with \(Sp(8, q)\) are \(\chi_{(21,1)}, \chi_{(1,21)}\), 2-cuspidal, also correspond to cuspidal unipotent characters of \(O^-(8, q)\). Here \(m_1 = 2, m_2 = 1, h_1 = 0, h_2 = \pm 2\).
$K$ a sufficiently large field of characteristic 0.

$L_n$ the category of quadratic unipotent representations of $G_n$ over $K$, $M_n$ the same for $H_n$.

**THEOREM** With the usual inner product, there is an isometry between $\bigoplus_{n \geq 0} K_0(L_n)$ and $\bigoplus_{n \geq 0} K_0(M_n)$.

Also: Both isomorphic to $\mathbb{Z}[N \times N] \times \bigoplus_{n,m \geq 0} K_0(\mathcal{H}_n \mod) \times K_0(\mathcal{H}_m \mod)$, $\mathcal{H}_n$ Hecke algebra of type $B_n$. 
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Theorem on unipotent blocks of $G = Sp(2n, q)$, $SO(2n + 1, q)$, $SO^{\pm}(2n, q)$ generalized to “quadratic unipotent” blocks.

EXAMPLE. $H_n = Sp(2n, q)$: quadratic unipotent characters in a block are constituents of $R^H_n(\psi)$,

$L \cong T_1 \times T_2 \times \ldots T_{M_1} \times T_1 \times T_2 \times \ldots T_{M_2} \times Sp(2m, q)$, where the $T_i$ are tori of order $q^f - 1$ and $\psi = 1 \times \mathcal{E} \times \chi_{\Lambda_1, \Lambda_2}$, $\Lambda_1$ and $\Lambda_2$ are $f$-cores.

Quadratic unipotent blocks classified by $e$-cores of pairs of symbols and weights.

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Quadratic unipotent blocks classified by $e$-cores of pairs of symbols and weights.
Fix an odd prime $\ell$, $e$ the order of $q \mod \ell$, $e = 2f$.

Let $f$ be odd. COMPARE:

$G_n = U(n, q)$ and $H_n = Sp(2n, q), \ q > n, \ \ell \text{ divides } q^f - 1$

$G_n = GL(n, q)$, and $H_n = Sp(2n, q), \ q > n,$

$\ell \text{ divides } q^f + 1$.

Also: $e = 2f$ where $f$ is even, i.e. $e \equiv 0 \pmod{4}$ and $\ell$ divides $q^f + 1$. Exclude $e \equiv 2 \pmod{4}$.
THEOREM Let $q, \ell$ be odd, and $q > n$. There are $\ell$-block correspondences between blocks $B$ of $G_n$ and blocks $b$ of $H_n$ as follows:

(i) $\ell$ divides $q^f - 1$, $f$ odd, $B$ a quadratic-unipotent $\ell$-block of $U(n, q)$ and $b$ a quadratic-unipotent $\ell$-block of $Sp(2m, q)$, some $m$

(ii) $\ell$ divides $q^f + 1$, $f$ odd, $B$ a quadratic-unipotent $\ell$-block of $GL(n, q)$ and $b$ a quadratic-unipotent $\ell$-block of $Sp(2m, q)$, some $m$

There is a natural bijection between quadratic-unipotent characters in $B$ and $b$.

When the defect groups are abelian, the defect groups are isomorphic and there is a perfect isometry between $B$ and $b$. 
Let $Bl(G_n)_{qu}$ (resp. $Bl(H_n)_{qu}$) be the set of quadratic unipotent blocks of $G_n$ (resp. $H_n$), $\ell$ divides $q^f - 1$ or $q^f + 1$ as above.

There is a bijection

$$\bigsqcup_{n \geq 0} Bl(G_n)_{qu} \leftrightarrow \bigsqcup_{n \geq 0} Bl(H_n)_{qu},$$

such that if $B \rightarrow b$, there is a natural bijection between quadratic-unipotent characters in $B$ and $b$. 
IDEA

Use the following correspondences:

\[ B \longleftrightarrow 2f - \text{core}(\lambda_1, \lambda_2) \longleftrightarrow \{(m_1, m_2, \rho_1, \rho_2)\} \longleftrightarrow \{(h_1, h_2, \rho_1, \rho_2)\} \longleftrightarrow f - \text{core}(\Lambda_1, \Lambda_2) \longleftrightarrow b \]

\[ m_1(m_1 + 1)/2 + m_2(m_2 + 1)/2 + 2N_1 + 2N_2 = n, \]
\[ h_1(h_1 + 1) + h_2^2 + N_1 + N_2 = m. \]
IDEA

Perfect Isometries “across types”:
Use the paper of [BMM] to get an isotopy from $B$ to a local subgroup of $G_n$ of the form $N_{G_n}(L, \lambda)$, then to a local subgroup of $H_n$, then to $b$. 
Enguehard has defined for a finite reductive group $G$, $s \in G^*$, a group $G(s)$ (can be called an endoscopy group).

Example: For $H_n = \text{Sp}(2n, q)$, $s$ with $s^2 = 1$, $H_n(s) = \text{Sp}(2m, q) \times \text{O}(2n - 2m, q)$.

We also have correspondences between unipotent blocks of suitable $G_n(s)$ and $H_n(s)$. 

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We also have correspondences between unipotent blocks of suitable $G_n(s)$ and $H_n(s)$. 
Here $Bl(G_n)_u$ denotes the set of unipotent blocks of $G_n$.

There is a bijection

\[
\bigsqcup_{n_1,n_2 \geq 0} Bl(G_{n_1} \times G_{n_2})_u \leftrightarrow \bigsqcup_{n_1,n_2 \geq 0} Bl(Sp_{2n_1} \times O_{2n_2})_u,
\]

such that if $B \rightarrow b$, there is a natural bijection between quadratic-unipotent characters in $B$ and $b$. 
SUMMARY

\[ \bigoplus_{n \geq 0} K_0(GL_n - \text{mod})_{qu} \cong \bigoplus_{n \geq 0} K_0(Sp_{2n} - \text{mod})_{qu}. \]

\[ \bigoplus_{n_1, n_2 \geq 0} K_0((GL_{n_1} \times GL_{n_2}) - \text{mod})_u \cong \bigoplus_{n_1, n_2 \geq 0} K_0((Sp_{2n_1} \times O_{2n_2}) - \text{mod})_u. \]
SUMMARY

For suitable \( \ell \): \( \bigsqcup_{n \geq 0} Bl(GL_n)_{qu} \leftrightarrow \bigsqcup_{n \geq 0} Bl(Sp_{2n})_{qu} \)

\( \bigsqcup_{n_1, n_2 \geq 0} Bl(G_{n_1} \times G_{n_2})_u \leftrightarrow \bigsqcup_{n_1, n_2 \geq 0} Bl(Sp_{2n_1} \times O_{2n_2})_u \)
What more can we say about this correspondence between blocks of general linear/unitary groups and blocks of symplectic groups? Are corresponding blocks derived equivalent? Morita equivalent?

Has the symplectic group reached equal status with the general linear group, her "all-embracing majesty"?
What more can we say about this correspondence between blocks of general linear/unitary groups and blocks of symplectic groups? Are corresponding blocks derived equivalent? Morita equivalent? Has the symplectic group reached equal status with the general linear group, her "all-embracing majesty"?


