Homework 1

Due Monday Feb. 2

Problem 1. Let P be a set of n points and L be a set of n curves in the plane such that every two members in L have at most *two* points in common. Show $|I(P, L)| \leq O(n^{3/2})$.

Problem 2. Let P be a set of n points in the plane, and let $G = (P, E)$ be the unit distance graph, i.e. edges are pairs of points with distance 1. Show that $|E(G)| \leq O(n^{4/3})$.

Problem 3. Show that n points in the plane determine at most $O(n^{7/3})$ triangles of unit area.

Problem 4. (a) Let P be an m-point set in the plane and let $k \leq \sqrt{m}$ be an integer parameter. Prove that at most $O(m^2/k)$ pairs of points of P lie on lines containing at least k and at most $\sqrt{m}$ points of P.

b) For $K \geq \sqrt{m}$, show that the number of pairs lying on lines with at least $\sqrt{m}$ and at most K points is $O(Km)$.

c) Prove the following: There is an absolute constant $c > 0$ such that for any n-point $P \subset \mathbb{R}^2$, at least cn^2 distinct lines are determined by P or there is a line containing at least cn points of P.

Problem 5. Consider a set K of n circles in the plane. Select a sample $S \subset K$ by s independent random draws with replacement. Consider the arrangement of S, and construct its vertical decomposition, that is, from each vertex extend vertical segments upwards and downwards until they hit a circle of S (or all the way to infinity). Similarly, extend vertical segments from the leftmost and rightmost points of each circle.

a) Show that this partitions the plane into $O(s^2)$ "trapezoids" (shapes bounded by at most two vertical segments and at most two circular arcs).

b) Show that for $s = Cr \ln n$ with a sufficiently large constant C, there is a positive probability that the sample S intersect all the dangerous interesting circular trapezoids, where "dangerous" and "interesting" are defined analogously to the definition in the proof of the weaker version of the cutting lemma.

Open problems. 1) Improve the upper bound in Problem 1.

2) Improve the upper bound in problem 2 (hard).

3) Can you prove a weak-cutting lemma statement for a family of pseudo segments, that is, a family of curves in the plane where every pair of curves intersect at most once. Such a statement would a have many applications.