Homework 1

Due Monday Feb. 2

Problem 1. Let P be a set of n points and L be a set of n curves in the plane such that every two members in L have at most *two* points in common. Show $|I(P, L)| \leq O(n^{3/2})$.

Solution. No three points lie on two curves by assumption. Therefore the incidence graph is $K_{3,2}$-free and the statement follows by applying the Kovari-Sos-Turan theorem.

Problem 2. Let P be a set of n points in the plane, and let $G = (P, E)$ be the unit distance graph, i.e. edges are pairs of points with distance 1. Show that $|E(G)| \leq O(n^{4/3})$.

Sketch solution. Let C be a family of n unit circles, whose centers are at each point of P. Then the maximum number of unit distances spanned by P is within a constant factor of the number of incidences between P and C. Therefore it suffices to bound the number of incidences between points and unit-circles in the plane. This can be done by modifying the proof of Szemeredi-Trotter given in class. However one must be careful that now the underlying graph drawn in the plane is a multi-graph: Pairs of vertices may have more than one edge. However by a degree of freedom argument, every pair of vertices has at most 2 edges. This is not a serious issue, and one can modify the proof of the crossing lemma, and show that n-vertex multi-graphs $G = (V, E)$ with edge multiplicity 2 satisfies $cr(G) \geq \Omega(|E|^3/n^2) - O(n)$. The rest of the proof is the same.

Problem 3. Show that n points in the plane determine at most $O(n^{7/3})$ triangles of unit area.

Solution. For any two points u, v, any third point w for which the triple uvw spans a unit area triangle, must lie on one of the two lines parallel to uv and at distance $2/||uv||$ from uv (Recall area of triangle is (1/2)(base)(height)). Now fix a point $u \in P$. Then for all each $v \in P - p$ we have the two parallel lines at distance $2/||uv||$ from the segment uv. This gives rise to $2(n-1)$ lines. Notice that these lines are all unique. Indeed any two lines with the same slope must have distinct y-intercepts. Hence by Szemeredi trotter, the number of unit triangles with points u is at most $O(n^{4/3})$. By repeating this argument over all $u \in P$ gives $O(n^{7/3})$ unit triangles.

Problem 4. (a) Let P be an m-point set in the plane and let $k \leq \sqrt{m}$ be an integer parameter. Prove that at most $O(m^2/k)$ pairs of points of P lie on lines containing at least k and at most $\sqrt{m}$ points of P.

Solution. For $k \leq \sqrt{m}$, by Szemeredi-Trotter, the number of k-rich lines is at most $O(m^2/k^3)$. Let i be an integer such that $k \leq 2^i \leq \sqrt{m}/2$. Then let L_i be the set of lines in the planes that contains at least 2^i points, but no more than 2^{i+1} points. By Szemeredi-Trotter, we have

$$|L_i| \leq O\left(\frac{m^2}{2^{3i}}\right).$$

and the number of pairs of points that lie on such lines is at most

$$O\left(\frac{m^2}{2^{3i}}\right) \cdot \binom{2^{i+1}}{2} = O\left(\frac{m^2}{2^i}\right).$$

Hence the number of pairs of points that lie on at least k points but no more than $\sqrt{m}$ points is at most

$$\sum_{i=\log k}^{\log(\sqrt{m})} O\left(\frac{m^2}{2^i}\right) = O(m^2/k)\,(1 + 1/2 + 1/4 + \cdots) = O(m^2/k)$$

b) For $K \geq \sqrt{m}$, show that the number of pairs lying on lines with at least $\sqrt{m}$ and at most K points is $O(Km)$.

Solution. For $K \geq \sqrt{m}$, the number of K-rich lines is at most m/K. Using this bound and following the argument above gives the statement.

c) Prove the following: There is an absolute constant $c > 0$ such that for any n-point $P \subset \mathbb{R}^2$, at least cn^2 distinct lines are determined by P of there is a line containing at least cn points of P.

Solution. Choose C sufficiently large such that the number of pairs of points that lie on a lie with at least 2^i points but no more than 2^{i+1}, for all i satisfying $C < 2^i < n/C$, is at most $n^2/4$. Therefore, at least $n^2/4$ pairs of points lie on lines with less than C points, or more than n/C points. If there is a pair that lies on n/C points, then we are done. Otherwise we have $n^2/4$ pairs that lie on lies with fewer than C points. Let L be all such lines. Then the number of distinct lines generated by our point set is at least $n^2/(4(C(C-1)/2))$, and the statement follows.

Problem 5. Consider a set K of n circles in the plane. Select a sample $S \subset K$ by s independent random draws with replacement. Consider the arrangement of S, and construct its vertical decomposition, that is, from each vertex extend vertical segments upwards and downwards until they hit a circle of S (or all the way to infinity). Similarly, extend vertical segments from the leftmost and rightmost points of each circle.

a) Show that this partitions the plane into $O(s^2)$ "trapezoids" (shapes bounded by at most two vertical segments and at most two circular arcs).

Solution. One can prove by induction that s circles partitions the plane into at most $O(s^2)$ regions, and creates at most $O(s^2)$ intersections (vertices). Since each vertex generates at most 1 additional region in the vertical decomposition process, we have at most $O(s^2) + O(s^2) = O(s^2)$ trapezoids.

b) Show that for $s = Cr \ln n$ with a sufficiently large constant C, there is a positive probability that the sample S intersect all the dangerous interesting circular trapezoids, where "dangerous" and "interesting" are defined analogously to the definition in the proof of the weaker version of the cutting lemma.

Solution. The proof follows the same argument given in lecture.

Open problems. 1) Improve the upper bound in Problem 1.

2) Improve the upper bound in problem 2 (hard).

3) Can you prove a weak-cutting lemma statement for a family of pseudo segments, that is, a family of curves in the plane where every pair of curves intersect at most once. Such a statement would a have many applications.