Homework 1

MCS 421 Combinatorics

Problem 1.1 Given a chess board with m rows and n columns, assume either m or n is even, say m. Then we can easily tile the chess board by tiling each columns one by one.

In the other direction, assume that our chess board has a perfect tiling. We need to show that either m or n is even. Assume for sake of contradiction that both n and m are odd. Then there are mn squares, which is odd, and therefore is impossible to tile and this contradicts our hypothesis.

Problem 1.4.a. The number of ways start tiling a $2 \times n$ chessboard, is to have 1 vertical domino (and then tile the rest $2 \times n - 1$ chessboard), or to have 2 horizontal dominoes (and then tile the rest $2 \times n - 2$ chessboard). Hence $f(n) = f(n-1) + f(n-2)$. Clearly $f(1) = 1)$ and $f(2) = 2$. It is now easy to compute $f(3), f(4), f(5)$.

Problem 1.4.b. By problem 1, we know $g(n) = 0$ if n is odd. Therefore we can assume n is even. To start off a perfect tiling, we look at the first two columns, and there are 3 ways to tile it (2 verticals on the bottom and 1 horizontal on top, 2 verticals on top and 1 horizontal on bottom, and 3 horizontals). Hence $g(n) = 3g(n-2)$. Clearly $g(2) = 3$ (by the pervious sentence), so $g(4) = 9$, $g(6) = 27$

Problem 1.38. (sketch) Let M be the matching that minimizes the length of all of the segments. Then there are no crossing segments, since otherwise we would contradict the triangle inequality (needs to be proven).

Problem 2.1. No further restrictions, there are 5^4 solutions. For (a) to hold we have $5 \cdot 4 \cdot 3 \cdot 2 = 120$ solutions. For (b) to hold we have $5^3 \cdots 2 = 250$ (last digit must be 2 or 4) solutions. For (a) and (b) to hold we have, last digit must be 2 or 4, then there are 4 choices for the next digit, 3 choices for the next, and 2 choices for the last giving us $2 \cdot 4 \cdot 3 \cdot 2 = 48$.

Problem 2.2. 13! ways to order each suit and there are 4! ways to order the "suit-blocks" to form a deck. Hence $(13!)^4 \cdot 4!$.

Problem 2.4. (a) Divisors for $3^4 \times 5^2 \times 7^6 \times 11$ are of the form $3^i \times 5^j \times 7^k \times 11^\ell$ where $0 \leq i \leq 4$, $0 \leq j \leq 2$, $0 \leq k \leq 6$, and $0 \leq \ell \leq 1$. Hence $5 \cdot 3 \cdot 7 \cdot 2$ divisors. (b) $620 = 2^2 \times 5 \times 31$. By the same analysis we have $3 \cdot 2 \cdot 2 = 12$ divisors. (c) $10^{10} = 2^{10} \times 5^{10}$, and we have $11 \cdot 11 = 121$ divisors.

Problem 2.7. In such an arrangement, there must be exactly 2 women between any two consecutive men. Number of ways to seat the men first is 3!. Number of ways to seat the remaining women is 8! (men are already sitting which implies there is an ordering). Thus the answer is $3! \times 8!$.