Homework 2

Due Monday, March 2

Problem 1. In the situation of Radon's Lemma, call a point $x \in \mathbb{R}^d$ a *Radon point* of A if it is contained in convex hulls of two disjoint subsets of A. Prove that if A is in general position (no $d+1$ members on a hyperplane), and $|A| = d+2$, then the Radon point is unique.

Problem 2. Given an n-point set $X \subset \mathbb{R}$ and a family F of $\alpha\binom{n}{2}$ X intervals (intervals with endpoints in X), there exists a point common to $\Omega(\alpha^2 n^2)$ intervals of F.

Problem 3. Show that any set of $\binom{2n-5}{n-2} + 2$ points in the plane in general position contains n members in convex position.

Problem 4. Show that any compact set in $\mathbb{R}^d$ has a unique point with the lexicographically smallest coordinate vector.

Problem 5. Let $\mathcal{C}_1, \ldots, \mathcal{C}_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$ such that for any choice of sets $C_1 \in \mathcal{C}_1, \ldots, C_{d+1} \in \mathcal{C}_{d+1}$, the intersection $C_1 \cap \cdots \cap C_{d+1}$ is nonempty. Then for some i all the sets of $\mathcal{C}_i$ have a nonempty intersection.

Problem 6. (a) Show that for n sufficiently large, any n-point set X in general position in the plane contains at least cn^{2k} convex subsets of size $2k$, where $c = c(k)$.

(b) Let $S = \{p_1, p_2, \ldots, p_{2k}\}$ be a convex subset of X, where the points are numbered in clockwise order along the boundary of their convex hull. The *holder* of S is the set $H(S) = \{p_1, p_3, \ldots, p_{2k-1}\}$. Show that there is a set H that is the holder of at least $\Omega(n^k)$ sets S.

(c) Infer the positive-fraction Erdős-Szekeres theorem in the plane.