Homework 5

MCS 421 Combinatorics

Problem 5.6. The coefficient of x^5y^{13} is $3^5(-2)^{13}\binom{18}{5}$. The coefficient of x^8y^9 is 0.

Problem 5.7. By the Binomial theorem with $x = 1$ and $y = 2$ we have the equation. Likewise for $x = 1$ and $y = k$ we have the summation equalling $(1 + r)^n$.

Problem 5.9. Using the Binomial Theorem, we have for n is even, the sum is 9^n. When n is odd the sum is -9^n.

Problem 5.15. For a variable x consider

$$(1-x)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k$$

taking the derivative on both sides with respect to x gives

$$-n(1-x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k k x^{k-1}.$$

set $x = 1$ give the desired equality.

Problem 5.18. By above we have

$$(x-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^k.$$

Integrating both sides with respect to x gives

$$\frac{(x-1)^{n+1}}{n+1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{x^{k+1}}{k+1} + C.$$

Setting $x = 0$ gives $C = (-1)^{n+1}/(n+1)$. Hence by setting $x = 1$ we have

$$\frac{(-1)^n}{n+1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \frac{1}{k+1}.$$

Problem 5.23. (a) $\binom{24}{14}$.
(b) $\binom{9}{4}\binom{15}{6}$.
(c) $\binom{9}{4}\binom{9}{3}\binom{6}{3}$.

(d) (b) - (c).

Problem 5.25. Let S be a set of size $m_1 + m_2$. Partition S into two sets A and B of size m_1 and m_2 respectively. The number of n-element subsets of S is $\binom{m_1+m_2}{n}$ by definition. We can count this in another way. For $k = 0, 1, \dots, n$, choose an n-element set from S with exactly k elements from A, and then $n - k$ elements from B. The number of ways to choose such an n-element set is $\binom{m_1}{k}\binom{m_2}{n-k}$ by the multiplication principle. Hence summing over all k gives

$$\binom{m_1 + m_2}{n} = \sum_{k=0}^{n} \binom{m_1}{k} \binom{m_2}{n-k}.$$