Homework 6

MCS 421 Combinatorics

Problem 6.1. Let A_1 be the set of integers between 1 and 10000 (inclusive) that is divisible by 4, A_2 be the integers divisible by 5, and A_3 divisibly by 6. Then $|A_1| = 2500$, $|A_2| = 2000$ and $|A_3| = 1666$. $|A_1 \cap A_2| = 500$, $|A_2 \cap A_3| = 333$ and $|A_1 \cap A_3| = 833$. Finally $|A_1 \cap A_2 \cap A_3| = 166$. By the inclusion exclusion principle we have $|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3| = 10000 - 2500 - 2000 - 1666 + 333 + 833 + 500 - 166 = 5334$.

Problem 6.6. We are looking for integral solution for $x + y + z = 12$ such that $0 \leq x \leq 6, 0 \leq y \leq 6, 0 \leq z \leq 3$. Let S denote all solution with $x \geq 0, y \geq 0, z \geq 0$. Number of such solution is $\binom{14}{2}$. Let A_1 be the set of these solutions with $x \geq 7$. Then $|A_1| = \binom{7}{2}$. Let A_2 be the set of solution with $y \geq 7$, then $|A_2| = \binom{7}{2}$. And let A_3 be the set of solutions with $z \geq 4$, which implies $|A_3| = \binom{10}{2}$. Hence $|A_1 \cap A_2| = 0$, $|A_2 \cap A_3| = \binom{3}{2}$ and $|A_1 \cap A_3| = \binom{3}{2}$. Finally $|A_1 \cap A_2 \cap A_3| = 0$. By the inclusion exclusion principle, our answer is

$$\binom{14}{2} - \binom{7}{2} - \binom{7}{2} - \binom{10}{2} + \binom{3}{2} + \binom{3}{2} = 10.$$

Problem 6.12. $\binom{8}{4} D_4$.

Problem 6.15. (a) D_7. (b) $7! - D_7$. (c) $7! - 7D_6 - D_7$.

Problem 6.17. Let S be the set of all permutations. Then we have $|S| = \frac{9!}{3!4!2!}$. Let A_1 be the number of these permutations with *aaa*. Then $|A_1| = \frac{7!}{4!2!}$. Let A_2 be the permutation with *bbbb*. Then we have $|A_2| = \frac{6!}{3!2!}$. Finally let A_3 be the permutations with *cc*. Then $|A_3| = \frac{8!}{3!4!}$. Hence $|A_1 \cap A_2| = \frac{4!}{2!}$, $|A_2 \cap A_3| = \frac{6!}{4!}$ and $|A_1 \cap A_3| = \frac{5!}{3!}$. Finally we have $|A_1 \cap A_2 \cap A_3| = 3$. By the inclusion exclusion principle, our answer is

$$\frac{9!}{3!4!2!} - \frac{7!}{4!2!} - \frac{6!}{3!2!} - \frac{8!}{3!4!} + \frac{4!}{2!} + \frac{6!}{4!} + \frac{5!}{3!} - 3$$

Problem 6.18. $(n-1)! = (n-1) \cdot (n-2)!$.

Problem 6.19.

$$D_n - (n-1)(D_{n-1} + D_{n-2}) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} - (n-1)(n-2)! \sum_{i=0}^{n-2} \frac{(-1)^i}{i!} - (n-1)(n-1)! \sum_{i=0}^{n-1} \frac{(-1)^i}{i!}.$$

combining like terms gives us 0.