Homework 7

MCS 421 Combinatorics

Problem 7.4. For $n \geq 5$ we have $f_n = f_{n-1} + f_{n-2}$, $f_{n-1} = f_{n-2} + f_{n-3}$, $f_{n-2} = f_{n-3} + f_{n-4}$, $f_{n-3} = f_{n-4} + f_{n-5}$. By a repeated substitution we have

$$f_n = 5f_{n-4} + 3f_{n-5}$$

. Hence f_n is divisibly by 5 if and only if f_{n-5} is divisible by 5. Together with the initial conditions, f_n is divisible by 5 if and only of n is divisible by 5.

Problem 7.8. Clearly $h_1 = 1$ and $h_2 = 2$. For $n \geq 3$, a proper coloring occurs in two cases. Case 1: we start with red. Since there are no two consecutive reds, then next square must be blue. Then the number of ways to color the rest is h_{n-2}. Case 2: we start with blue. Then to color the remaining board, we have h_{n-1} solutions. Hence $h_n = h_{n-1} + h_{n-2}$.

Problem 7.9. $h_0 = 1$ since there is only one way to tile the empty board (do nothing). $h_1 = 3$ by inspection. For $n \geq 2$, we have 2 cases. Case 1: Start with blue or white. Then the number of ways to tile the rest is h_{n-1}, giving us $2h_{n-1}$ solutions. Case 2. Start with red, then color must be blue or white, giving us $2h_{n-2}$ way to color the rest. Hence $h_n = 2h_{n-1} + 2h_{n-2}$. In order to get the closed form, we solve $x^2 - 2x - 2 = 0$, giving us roots at $x = 1 + \sqrt{3}$ and $x = 1 - \sqrt{3}$. There for the solution to the recurrence is

$$h_n = a(1+\sqrt{3})^n + b(1-\sqrt{n})^n$$

Now in order to satisfy the initial conditions, we have for $n = 0$, $1 = a + b$, and $n = 1$ gives $3 = a(1+\sqrt{3}) + b(1-\sqrt{3})$. Solving these equations gives $a = \frac{\sqrt{3}+2}{2\sqrt{3}}$ and $b = \frac{\sqrt{3}-2}{2\sqrt{3}}$. Hence

$$h_n = \frac{\sqrt{3}+2}{2\sqrt{3}}(1+\sqrt{3})^n + \frac{\sqrt{3}-2}{2\sqrt{3}}(1-\sqrt{3})^n.$$

Problem 7.11. (a) Define the recurrence $a_n = l_n - f_{n-1} - f_{n+1}$. Hence $a_1 = a_2 = 0$, and $a_n = a_{n-1} + a_{n-2}$. Therefore $a_n = 0$, which implies the result. (b) We proceed by induction on n. The base case $n = 0$ follows by obtaining 4. For the inductive step, we assume the statement is true for $n' < n$, and try to prove the it for n. Hence

$$l_0^2 + \cdots + l_n^2 = l_{n-1}l_n + 2 + l_n^2 = l_n(l_{n-1} + l_n) + 2 = l_n l_{n+1} + 2.$$

Problem 7.13. (a) $(1-cx)^{-1}$; (b) $(1+x)^{-1}$; (c) $(1-x)^{\alpha}$; (d) e^x; (e) e^{-x}.

Problem 7.14. (a) $(x + x^3 + x^5 + \cdots)^4 = x^4(1 - x^2)^{-4}$. (b) $(1 + x^3 + \cdots)^4 = (1 - x^3)^{-4}$. (c) $(1 + x)(1 + x + x^2 + \cdots)^2 = (1 + x)(1 - x)^{-2}$. (d) $x^3(1 + x^2 + x^10)(1 + x^2 + x^3)(1 - x)^{-2}$ (e) $x^40(1 - x)^{-4}$

Problem 7.16. h_n is the number of n-combinations in the multi-set $e_1, ..., e_4$, each element occuring ∞ number of times such that e_1 appears at most twice, e_2 is even and at most 6, e_3 is even, and e_4 is not zero.

Problem 7.22. $g(x) = \sum\limits_{n=0}^{\infty} n! \frac{x^n}{n!} \frac{x^n}{n!} = \frac{1}{1-x}$.