Homework 9

MCS 421 Combinatorics

Problem 7.33. The characteristic polynomial is

$$x^3 - x^2 - 9x + 9 = (x-3)(x+3)(x-1).$$

Hence the general solution is $h_n = a3^n + b(-3)^n + c$. Using the initial values gives $a = 1/3$, $b = -1/12$ and $c = -1/4$.

Problem 7.34. The characteristic polynomial is $x^2 - 8x + 16 = (x-4)^2$. Therefore the general solution is $h_n = a4^n + bn4^n$. Using the initial values gives $a = -1$ and $b = 1$.

Problem 7.37. The number of ternary strings starting with 22 is a_{n-2}. Now lets count of of the others. We claim that for any ternary string of length $n-1$, there are two ways to extend it to a ternary string of length n, which is not starting with 22. Indeed if the ternary string of length $n-1$ starts with a 0, then it can be extended by adding 1 or 2 before it. If the ternary string of length $n-1$ starts with a 1, then it can be extended by adding 0 or 2 before it. If the ternary string of length $n-1$ starts with a 2, then it can be extended by adding 0 or 1 before it (22 is already counted). Hence $a_n = 2a_{n-1} + a_{n-2}$.

Problem 7.40. The number of ternary strings starting with 2 is clearly a_{n-1}. In order to count the rest, we will start with a ternary string of length $n-2$, and show that there are 2 ways to extend it. Indeed if the $(n-2)$-length ternary string starts with a 0, then the digit to its left must be a 2, and then the digit before that must be 0 or 1 giving it two ways to extend. If the $(n-2)$-length ternary string starts with a 1, then the digit to its left must be a 2, and then the digit before that must be 0 or 1 giving it two ways to extend. If the $(n-2)$-length ternary string starts with a 2, then (with some checking) the digit to its left must be a 2, and then the digit before that must be 0 or 1 giving it two ways to extend. Hence $a_n = a_{n-1} + 2a_{n-2}$. This gives the characteristic polynomial $x^2 - x - 2 = (x-2)(x+1)$, and therefore the general solution is

$$a_n = a2^n + b(-1)^n.$$

Using initial values gives $a = 4/3$ and $b = -1/3$.

Problem 7.42. We have $h_0 = 3$ and $h_1 = 16$. Hence

$$h_n - 8h_{n-1} + 16h_{n-2} = 0.$$

the characteristic polynomial is $x^2 - 8x + 16 = (x-4)^2$. Therefore the general solution is $h_n = an4^n + b4^n$. Using the initial values gives $a = 1$ and $b = 3$.

Problem 7.43. We have $h_0 = 1$ and $h_1 = 10$. Hence $h_n - 16h_{n-1} + 8h_{n-2} = 0$. The characteristic polynomial is $x^2 - 6x + 8 = (x-4)(x-2)$. Therefore the general solution is $h_n = a4^n + b2^n$. Using initial values gives $a = 4$ and $b = -3$.