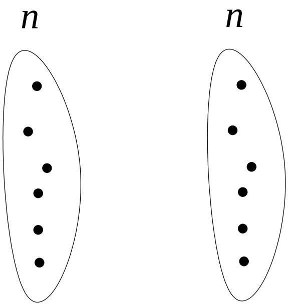
Extremal results for Semi-algebraic hypergraphs

Andrew Suk (UIC)

November 4, 2015

A classic theorem of Kövári, Sós, and Turán

Bipartite graph G, edge set E. $n \to \infty$



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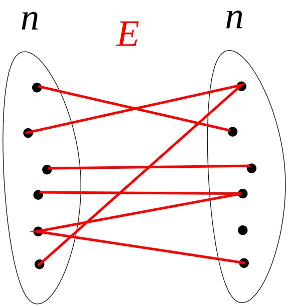

Number of edges $|E| \leq n^2$.

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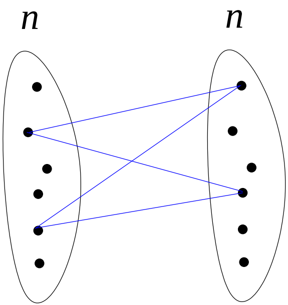

Assumption: No $K_{2,2}$ as a subgraph.

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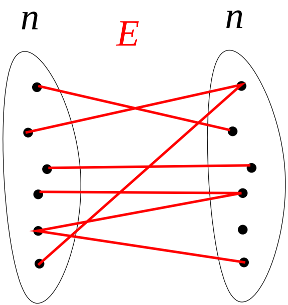
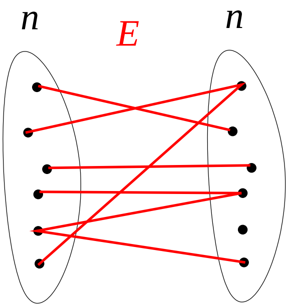


Theorem (Kövári, Sós, and Turán)

If G is $K_{2,2}$-free, then $|E(G)| \leq O(n^{3/2})$.

A classic theorem of Kövári, Sós, and Turán

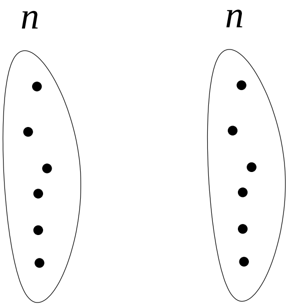


Theorem (Kövári, Sós, and Turán)

If G is $K_{t,t}$-free, $t \geq 2$, then $|E(G)| \leq cn^{2-1/t} + tn.$

A classic theorem of Kövári, Sós, and Turán

Either $|E| \geq n^2/2$ or $|\overline{E}| \geq n^2/2$.



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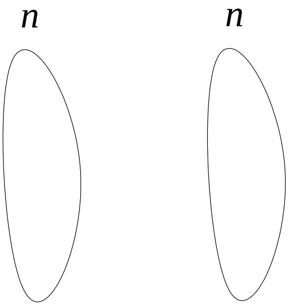
A Ramsey-type result

Theorem (Kövári, Sós, and Turán)

If $G = (V_1, V_2, E)$ is a bipartite graph with $|V_1|, |V_2| = n$, then there are subsets $U_1 \subset V_1, U_2 \subset V_2$ such that $|U_1| = |U_2| = c' \log n$, and either $U_1 \times U_2 \subset E$ or $U_1 \times U_2 \cap E = \emptyset$.



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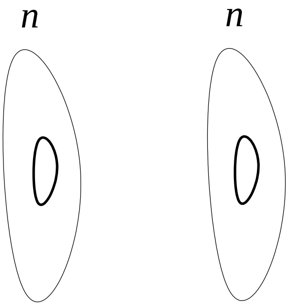

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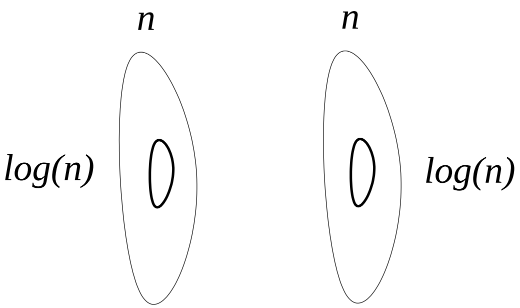

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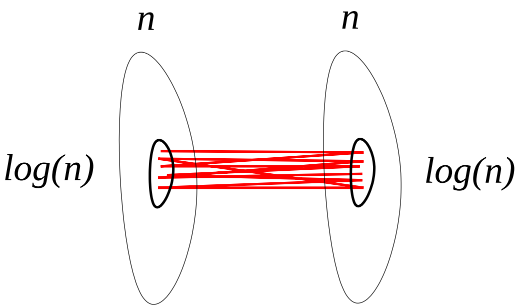

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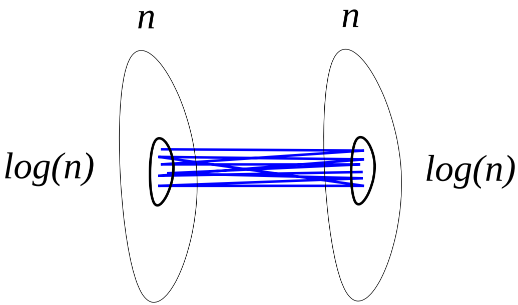
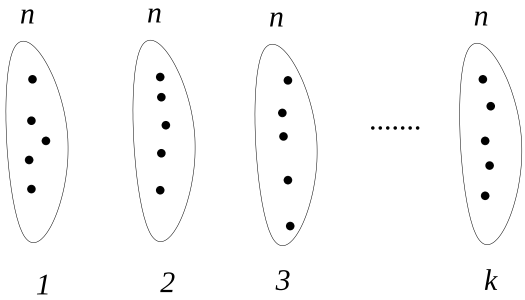


Result is tight by a random construction.

Generalized by Erdős

k-partite k-uniform hypergraph $H = (V, E)$, $|V| = kn$.

k is fixed and $n \to \infty$



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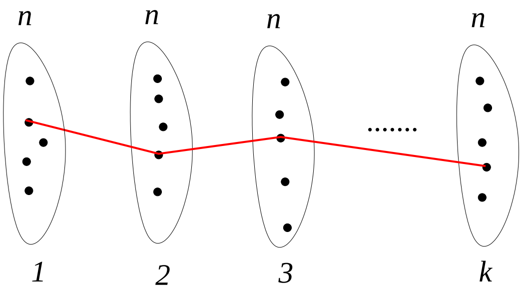

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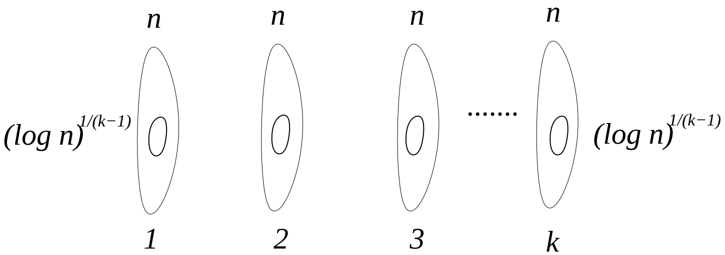

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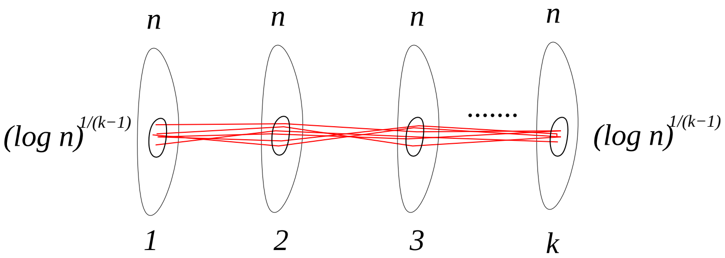
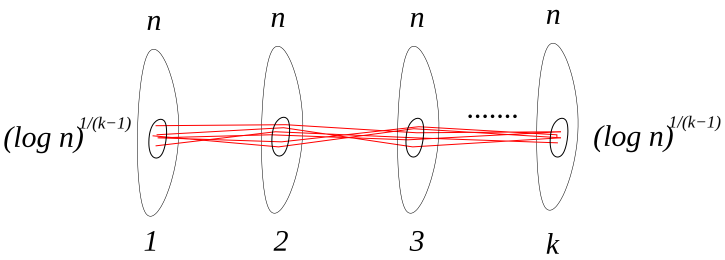


These results are tight.

Generalization, Erdős

k-partite k-uniform hypergraph H, edge set E.



In this talk: We can improve these results if our graph or hypergraph is defined algebraically with low complexity. **Semi-algebraic** hypergraphs.

Generalization, Erdős

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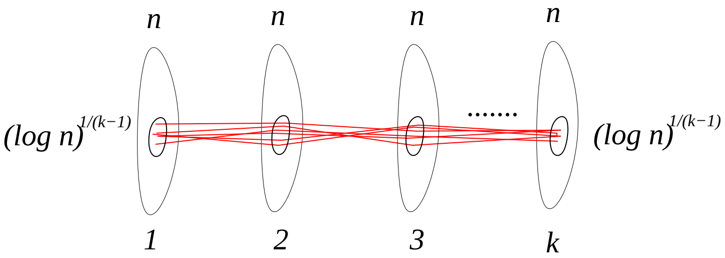
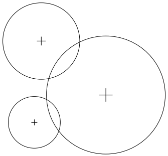


Semi-algebraic hypergraphs: $V = \{$simple geometric objects in $\mathbb{R}^d\}$, $E = \{$simple relation on k tuples of $V\}$.

$V = \{A_1, ..., A_n\}$, n disks in the plane. $E = \{$pairs of disks that intersect$\}$.



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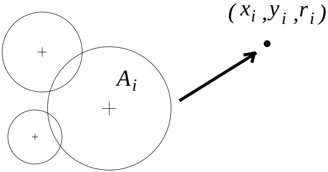

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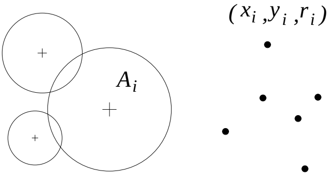

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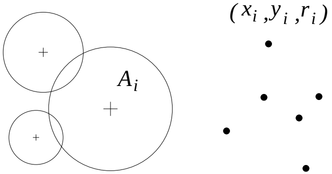

$A_i \rightarrow p_i = (x_i, y_i, r_i)$, $A_j \rightarrow p_j = (x_j, y_j, r_j)$. A_i and A_j cross if and only if

$$-x_i^2 + 2x_ix_j - x_j^2 - y_i^2 + 2y_iy_j - y_j^2 + r_i^2 + 2r_ir_j + r_j^2 \geq 0.$$

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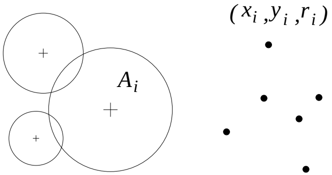


Graph $G = (V, E)$, $V = n$ points in $\mathbb{R}^3$

E defined by the polynomial

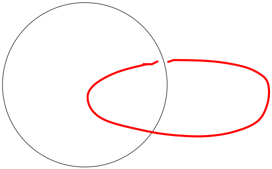
$$f(z_1, ..., z_6) = -z_1^2 + 2z_1z_4 - z_4^2 - z_2^2 + 2z_2z_5 - z_5^2 + z_3^2 + 2z_3z_6 + z_6^2.$$

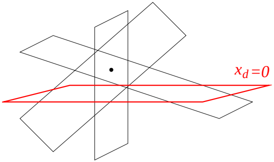
$(p_i, p_j) \in E \Leftrightarrow f(p_i, p_j) \geq 0.$

Examples

1. $V = \{n$ circles in $\mathbb{R}^3\}$
   $E = \{$pairs that are linked$\}$.

2. $V = \{n$ hyperplanes in $\mathbb{R}^d$ in general position$\}$,
   $E = \{d$-tuples whose intersection point is above the hyperplane $x_d = 0\}$.





More examples of semi-algebraic hypergraphs

Examples

1. $V = \{n \text{ circles in } \mathbb{R}^3\} \rightarrow$ n points in higher dimensions.
   $E = \{\text{pairs that are linked}\} \rightarrow$ polynomials $f_1, ..., f_t$.

2. $V = \{n \text{ hyperplanes in } \mathbb{R}^d \text{ in general position}\} \rightarrow$ n points in higher dimensions,
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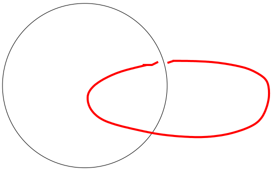

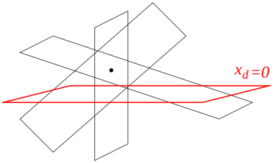

We say that $H = (V, E)$ is a **semi-algebraic k-uniform hypergraph in d-space** if

$V = \{n \text{ points in } \mathbb{R}^d\}$

E defined by polynomials $f_1, \ldots, f_t$ and a Boolean formula Φ such that

$$(p_{i_1}, \ldots, p_{i_k}) \in E$$

$$\Leftrightarrow \Phi(f_1(p_{i_1}, \ldots, p_{i_k}) \geq 0, \ldots, f_t(p_{i_1}, \ldots, p_{i_k}) \geq 0) = \text{yes}$$

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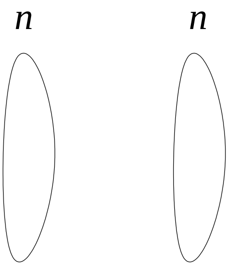
$n \to \infty$

E **has bounded complexity:** k = uniformity, d = dimension, t, and $deg(f_i)$ is bounded by some constant. (say ≤ 1000).

Previous results

Theorem (Alon, Pach, Pinchasi, Radoicic, Sharir 2005)

Let $H = (V_1, V_2, E)$ be a bipartite semi-algebraic graph ($k = 2$) in d-space, where $|V_1| = |V_2| = n$. Then there are subsets $V_1', V_2' \subset V$ such that $|V_i'| \geq \epsilon n$ and either $(V_1', V_2') \subset E$ or $(V_1', V_2') \subset \overline{E}$, and $\epsilon = \epsilon(E)$.



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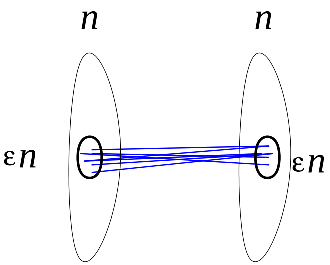

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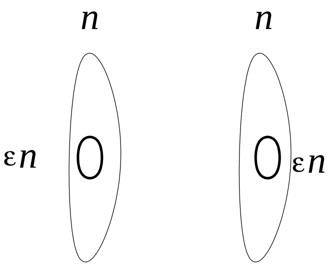

Stronger density theorem

Including an argument of Komlos:

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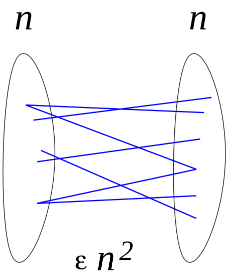

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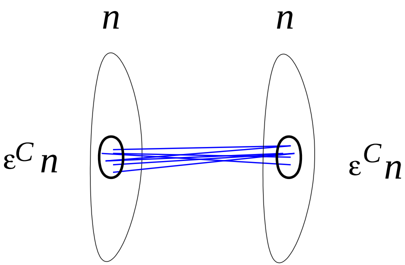
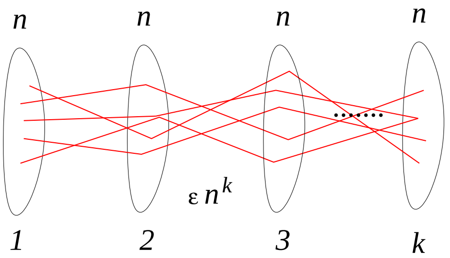


Generalization

Theorem (Fox, Gromov, Lafforgue, Naor, Pach 2012, Bukh and Hubard 2012)

Let $H = (V_1, ..., V_k, E)$ be a k-partite semi-algebraic k-uniform hypergraph in d-space, where $|V_1| = \cdots = |V_k| = n$. If $|E| \geq \epsilon n^k$, then there are subsets $V_1', ..., V_k' \subset V$ such that $|V_i'| \geq \epsilon^C n$ where $C = C(E)$, and $(V_1', ..., V_k') \subset E$.



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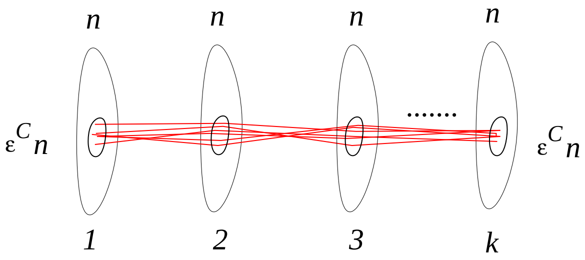
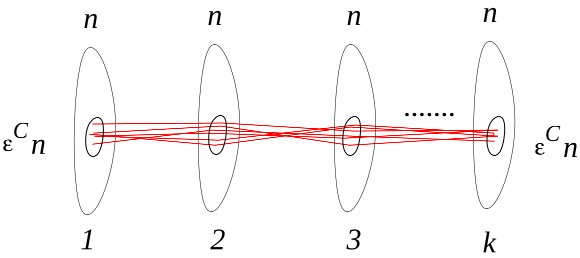


Generalization

$C(E) = C(k, d, f_1, \ldots, f_t)$: Dependency on uniformity k and dimension d is very bad.

Fox, Gromov, Lafforgue, Naor, Pach: $C \sim \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^d}}}}}}_{k}$ (tower-type)

Bukh-Hubard: $C(k, d, t, f_1, \ldots, f_t) \sim 2^{2^{k+d}}$, double exponential in $k + d$.



Bukh-Hubard: Set sizes decay triple exponentially in k

Theorem (Fox, Pach, S., 2015)

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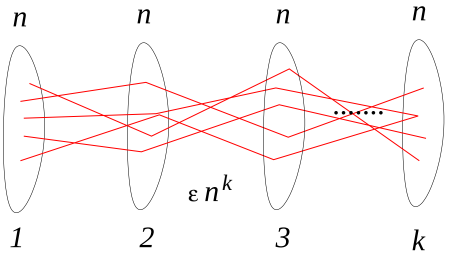

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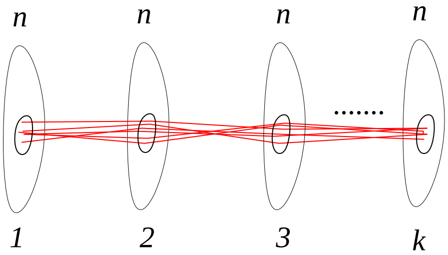


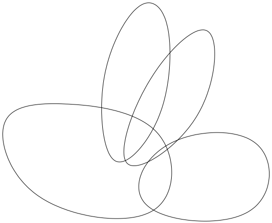
Applications, Tverberg-type result

Theorem (Pach, 1998)

Let $P_1, P_2, ..., P_{d+1} \subset \mathbb{R}^d$ be disjoint n-element point sets with $P_1 \cup \cdots \cup P_{d+1}$ in general position. Then there is a point $q \in \mathbb{R}^d$ and subsets $P'_1 \subset P_1, ..., P'_{d+1} \subset P_{d+1}$, with

$$|P'_i| \geq 2^{-2^{2^{O(d)}}} n,$$

such that all closed rainbow simplices contains q.



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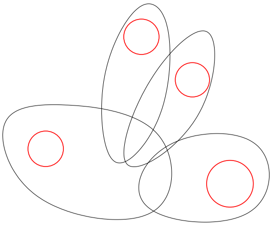

Applications, Tverberg-type result

Theorem (Pach, 1998)

Let $P_1, P_2, \ldots, P_{d+1} \subset \mathbb{R}^d$ be disjoint n-element point sets with $P_1 \cup \cdots \cup P_{d+1}$ in general position. Then there is a point $q \in \mathbb{R}^d$ and subsets $P_1' \subset P_1, \ldots, P_{d+1}' \subset P_{d+1}$, with

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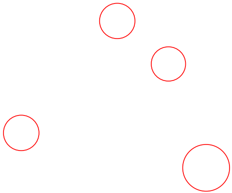

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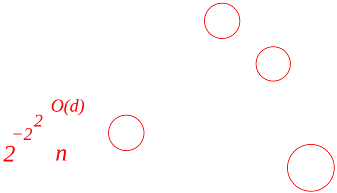

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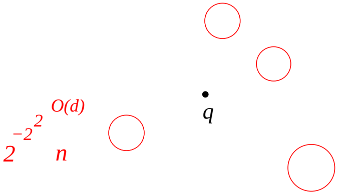

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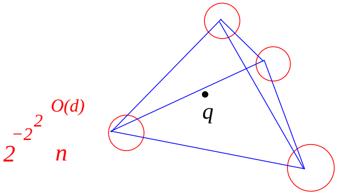

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Theorem (Karasev, Kynčl, Paták, Patáková, Tancer, 2015)

$$|P_i'| \leq 2^{-cd \log d} n,$$

Applications, Same-type Lemma

Theorem (Bárány and Valtr, 1998)

Let $P_1, \ldots, P_k$ be n-element point sets in $\mathbb{R}^d$ such that $P_1 \cup \cdots \cup P_k$ is in general position. Then there are subsets $P'_1 \subset P_1, \ldots, P'_k \subset P_k$ such that the k-tuple $(P'_1, \ldots, P'_k)$ has same-type transversals and

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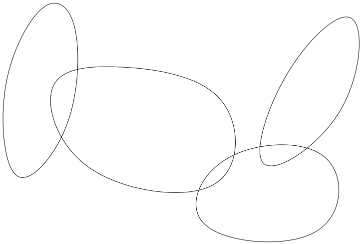

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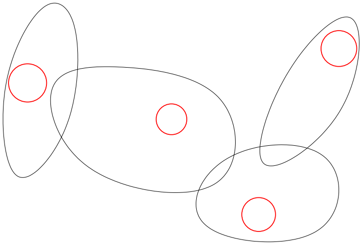

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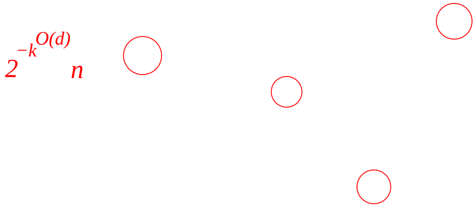

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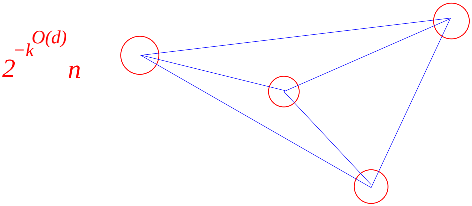

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Regularity lemma: Semi-algebraic k-uniform hypergraph $H = (V, E)$.

Theorem (Fox, Pach, S., 2015)

For any $\epsilon > 0$, we can partition V into at most $M(\epsilon)$ parts, such that almost all k-tuples of parts are **complete or empty**. *Moreover $M(\epsilon) < (1/\epsilon)^c$, where c depends only on k, d, E.*

Usual regularity: almost all k-tuples of parts are "**random**". $M(\epsilon)$ is huge:

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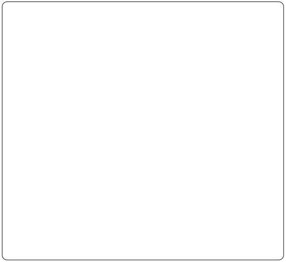

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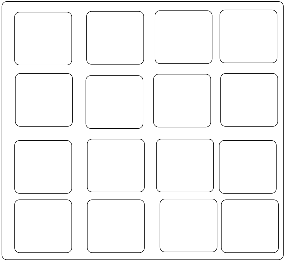

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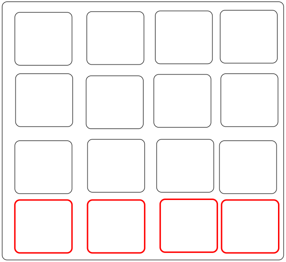
$H =$ 

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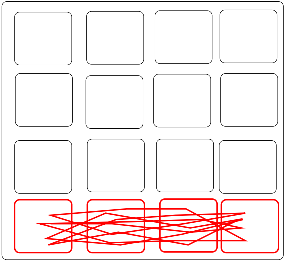
$H =$ 

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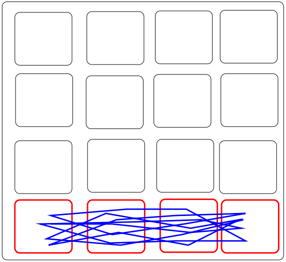
$H =$ 

Regularity lemma: Semi-algebraic k-uniform hypergraph in $\mathbb{R}^d$.

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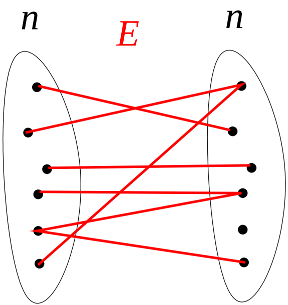
- $k = 2$, $M(\epsilon) \leq tower(1/\epsilon) = 2^{2^{2^{\cdots 2}}}$
- $k = 3$, $M(\epsilon) \leq wowzer(1/\epsilon) = tower(tower(...(tower(2))))$
- $k = 4$, $M(\epsilon) \leq wowzer(wowzer(...(wowzer(2))))$.

A classic theorem of Kövári, Sós, and Turán



Theorem (Kövári, Sós, and Turán)

If G is $K_{t,t}$-free, $t \geq 2$, then $|E(G)| \leq cn^{2-1/t} + tn$.

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If G is $K_{t,t}$-free, $t \geq 2$, then $|E(G)| \leq cn^{2-1/t} + tn$.

Conjecture

The above bound is tight.

(Brown 1966) $K_{2,2}$-free graph with $|E(G)| \geq cn^{3/2}$

(Brown 1966) $K_{3,3}$-free with $|E(G)| \geq cn^{5/3}$

Open for $t \geq 4$

Random construction: $K_{t,t}$-free with $|E(G)| \geq c_t n^{2-2/t}$.

A classic theorem of Kövári, Sós, and Turán

Theorem (Kövári, Sós, and Turán)

If G is $K_{t,t}$-free, $t \geq 2$, then $|E(G)| \leq cn^{2-1/t} + tn$.

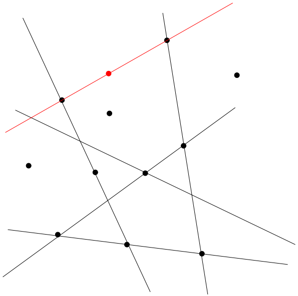
Problem: Can we improve the Kövári-Sós-Turán bound for semi-algebraic graphs?

Incidences between points and lines

Problem (Erdős)

Determine the maximum number of incidences between n points and n lines in the plane.

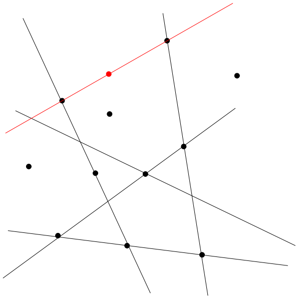
Incidence: (p, ℓ) such that $p \in \ell$.
$I(P, L) = \{(p, \ell) \in P \times L : p \in \ell\}$.

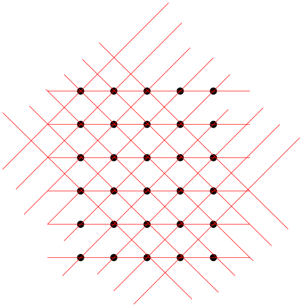


Incidences between points and lines

Theorem (Szemerédi-Trotter)

Let P be a set of n points in the plane and L a set of n lines in the plane. Then $|I(P, L)| \leq O(n^{4/3})$.





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Theorem (Szemeredi-Trotter)

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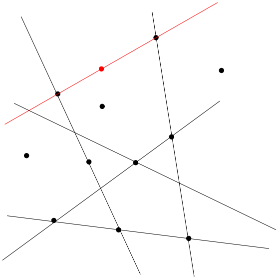
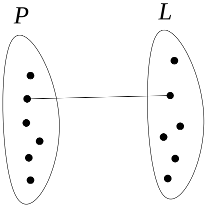
$L \rightarrow L^* = n$ points in $\mathbb{R}^2$. $K_{2,2}$-free

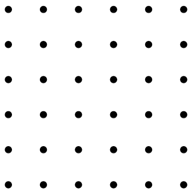




Unit distance problem

Problem

Determine the maximum number of unit distance pairs among n points in the plane.

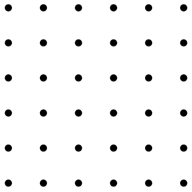


Unit distance problem

Theorem (Erdős, Spencer-Szemerédi-Trotter)

Let $u_2(n)$ denote the maximum number unit distances that can be spanned by n points in the plane.

$$n^{1+c/\log\log n} \leq u_2(n) \leq c' n^{4/3}$$



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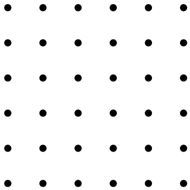
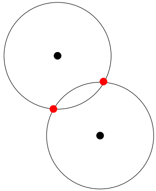
Semi-algebraic graph in the plane, $K_{2,3}$-free





Theorem (Fox, Pach, Sheffer, S., Zahl)

Let d and t be fixed, and let $G = (V_1, V_2, E)$ be a bipartite semi-algebraic graph in $\mathbb{R}^d$. If G is $K_{t,t}$-free, then

$$|E(G)| \leq O(n^{4/3}) \qquad d = 2$$

$$|E(G)| \leq O(n^{\frac{2d}{d+1}+o(1)}) \qquad d \geq 3.$$

Theorem (Kövári, Sós, and Turán)

If G is $K_{t,t}$-free, $t \geq 2$, then $|E(G)| \leq cn^{2-1/t} + tn$.

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Corollary

$P = \{n \text{ points in the plane}\}$
$L = \{n \text{ strips in the plane with unit width}\}$

No t points in P lie inside t strips of L, then

$$|I(P, L)| \leq O(n^{4/3}).$$

Unit distance problem in higher dimensions

Theorem (Erdős-Pach)

Let $u_d(n)$ denote the maximum number of times the unit distance can occur among n points in $\mathbb{R}^d$. For $d \geq 4$,

$$u_d(n) = \Theta(n^2).$$

Orthogonal circles: $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$

$P_1 = \{(x_1, x_2, 0, 0), n$ points on the circle $x_1^2 + x_2^2 = 1/2\}$
$P_2 = \{(0, 0, x_3, x_4), n$ points on the circle $x_3^2 + x_4^2 = 1/2\}$

Forbid $K_{t,t}$: (Oberlin-Oberlin) $O(n^{7/4})$, plus some strong conditions.

Theorem (Fox, Pach, Sheffer, S., Zahl)

Let d and t be fixed, and let $G = (V_1, V_2, E)$ be a bipartite semi-algebraic graph in $\mathbb{R}^d$. If G is $K_{t,t}$-free, then

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Corollary

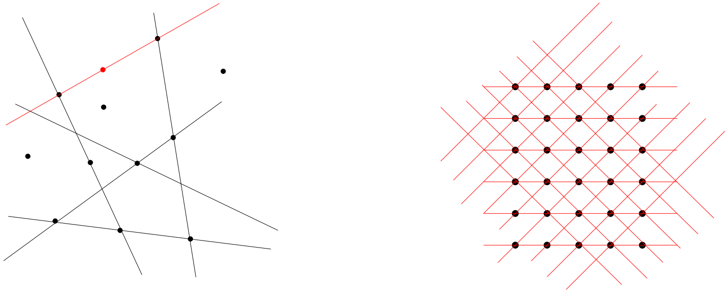
For fixed $t > 0$, let P be a set of n points in $\mathbb{R}^4$ such that there are no two "orthogonal circles" with t points on each circle. Then the maximum number of unit distances spanned by P is at most $O(n^{8/5+o(1)})$.

Oberlin-Oberlin: $O(n^{7/4})$, under much stronger conditions.

Open problems

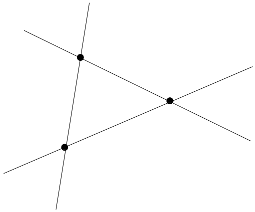
Theorem (Szemeredi-Trotter)

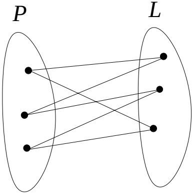
Let P be a set of n points in the plane and L a set of n lines in the plane. Then $|I(P, L)| \leq O(n^{4/3})$.



Open Problems

F_1 =

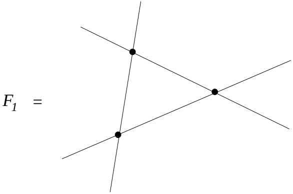


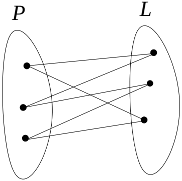


Problem

Let P be a set of n points and L be a set of n lines in the plane. If $|I(P, L)| \geq \Omega(n^{4/3})$ incidences, then does $P \cup L$ contain F_1?

Open Problems





Conjecture

$P \cup L$ is F_1-free, then $|I(P, L)| \leq O(n^{4/3-\epsilon})$.

(Solymosi 2005) $n^{4/3} / \log^* n$

Open Problems

Theorem (Erdős)

Every n-vertex graph with no C_6 has at most $O(n^{4/3})$ edges.

Conjecture

Let G be an n-vertex semi-algebraic bipartite graph in $\mathbb{R}^d$ with no C_6. Then $|E| \leq o(n^{4/3})$.

Open Problems

Theorem (Erdős)

Every n-vertex graph with no C_{2k} has at most $O(n^{1+1/k})$ edges.

Conjecture

Let G be an n-vertex semi-algebraic graph in $\mathbb{R}^d$ with no C_{2k}. Then $|E| \leq o(n^{1+1/k})$.

Thank you!