Lecture 13

1 Monotone paths in ordered hypergraphs

Given a k-uniform hypergraph $H = (V, E)$, where $V = \{1, \ldots, N\}$, a monotone path in H of length n is a set of n vertices $v_{i_1}, \ldots, v_{i_n}$ such that $i_1 < \cdots < i_n$ and $(v_{i_x}, v_{i_{x+1}}, v_{i_{x+2}}) \in E(H)$ for all $x = 1, \ldots, n-2$. Let $N_k(q, n)$ be the minimum N such that for any q-coloring on the edges of a complete N-vertex k-uniform hypergraph $H = (V, E)$ where $V = \{1, \ldots, N\}$, contains a monochromatic monotone path of length n.

Theorem 1.1. *For graphs, we have $N_2(q, n) \leq (n-1)^q + 1$.*

Proof. Let $N = (n-1)^q + 1$ and let G be a complete graph where $V(G) = [N]$, and let χ be a q-coloring on the edges. We color each vertex $v_i \in V(G)$ with the q-tuple $(c_1, \ldots, c_q)$, where c_j denotes the longest j-colored monotone path ending at vertex v_i. For sake of contradiction, assume there is no monochromatic monotone path of length n. Then there are at most $(n-1)^q$ distinct q-tuples $(c_1, \ldots, c_q)$. By the pigeonhole principle, there are two vertices with the same color (vector), say vertex v_i and v_j where $i < j$. WLOG we can assume $v_i v_j$ has color 1. Then the longest 1-colored path ending in vertex v_i can be extended to a 1-colored path ending in v_j with length $c_1 + 1$, which is a contradiction.

□

Theorem 1.2. *For graphs, we have $N_3(q, n) \leq (n-1)^{n^{q-1}} + 1$.*

Proof. Let $N = (n-1)^{n^{q-1}} + 1$ and let H be complete N-vertex 3-uniform hypergraph where $V(H) = [N]$. Let χ be a q-coloring on the edges in H. We now define a coloring on the pairs of vertices. We color the pair $v_i v_j$ with the $(q-1)$-vector $(c_1, \ldots, c_{q-1})$, if the i-the colored path monotone path ending in vertices v_i, v_j has length c_i. If there is a monotone monochromatic path in one of the first $q-1$ colors, we are done. Hence we can assume no such path exist, and lets try to find a monotone path of length n in the last color q. By the assumption, there are at most $(n-1)^{q-1}$ colors distinct colors used for the pairs. By the theorem above, there are n vertices, say $v_{i_1}, ..., v_{i_n}$ such that every pair $v_{i_x}, v_{i_{x+1}}$ has color $(c_1, \ldots, c_{q-1})$. We claim these vertices induces a q-colored monotone path of length n. Indeed suppose for sake of contradiction there is a triple $v_{i_x}, v_{i_{x+1}}, v_{i_{x+1}}$ that has color j where $j < q$. Since the longest j-colored path ending in vertices $v_{i_x}, v_{i_{x+1}}$ has length c_j, the longest j colored path ending in $v_{i_x}, v_{i_{x+1}}, v_{i_{x+2}}$ has length $c_j + 1$, since it can be extended. Hence we have a contradiction, and this completes the proof.

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