Lecture 1

1 Point-line incidences

Let P be a set of m (distinct) points and L be a set of n (distinct) lines in the plane. An incidence is a pair $(p, \ell) \in P \times L$ such that $p \in \ell$. Denote $I(P, L) = \{(p, \ell) \in P \times L : p \in \ell\}$ and denote $I(m, n)$ to be the maximum $|I(P, L)|$ over all choices of an m-element point set P and an n-element set of lines L.

Trivial upper bound, $I(m, n) \leq mn$.

Theorem 1.1 (Kővári-Sós-Turán, 1954). *If $G = (U, V, E)$ is a bipartite graphs with parts of size m and n, ($|U| = m$ and $|V| = N$) not containing $K_{r,r}$ as a subgraph, then*

$$|E(G)| \leq O(\min(mn^{1-1/r} + n, m^{1-1/r}n + m)).$$

Proof.

$$m\binom{\frac{|E(G)|}{m}}{r} = m\binom{\frac{\sum_u d(u)}{m}}{r} \overset{Jenson}{\leq} \sum_{u \in U} \binom{d(u)}{r} = [\text{copies of } K_{1,r}] \overset{K_{r,r}-free}{\leq} \binom{n}{r}(r-1)$$

since $\frac{(n-r)^r}{r!} \leq \binom{n}{r} \leq \frac{n^r}{r!}$, we have

$$m\frac{\left(\frac{|E(G)|}{m} - r\right)^r}{r!} \leq \frac{n^r}{r!}r,$$

which implies

$$|E(G)| \leq r^{1/r}nm^{1-1/r} + rm.$$

The other bound, $|E(G)| \leq O(mn^{1-1/r} + n)$, follows from double counting $\sharp K_{r,1}$-s.

$\square$

No two distinct points lie on two distinct lines ($K_{2,2}$-free), implies

$$I(n, n) \overset{KST}{\leq} O(n^{3/2}).$$

Theorem 1.2 (Szemerédi-Trotter, 1983). *For all $m, n \geq 1$*

$$I(m, n) = O(m^{2/3}n^{2/3} + m + n)$$

and this bound is tight.

Corollary 1.3 (Szemerédi-Trotter, 1983)**.** *Given a set of m points P in the plane, and set of n k-rich lines L_k (i.e. each line in L_k contains at least k points from P), then*

$$n = |L_k| \leq O\left(\frac{m^2}{k^3} + \frac{m}{k}\right).$$

Proof.

$$kn \leq |I(P, L_k)| \overset{ST}{\leq} O(n^{2/3}m^{2/3} + m + n).$$

$\square$

References

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