Lecture 2

1 Point-line incidences

Let P be a set of m (distinct) points and L be a set of n (distinct) lines in the plane. An incidence is a pair $(p, \ell) \in P \times L$ such that $p \in \ell$. Denote $I(P, L) = \{(p, \ell) \in P \times L : p \in \ell\}$ and denote $I(m, n)$ to be the maximum $|I(P, L)|$ over all choices of an m-element point set P and an n-element set of lines L.

1.1 Upper bound proof of Székely, 1997

A *drawing* of a graph G is a mapping that assigns vertices to points, and edges to continuous arcs connecting the corresponding vertices. The interior of each arc (edge) does not contain any vertices. A pair of arcs (edges) cross if their interiors have a point in common. The *crossing number* of a graph G, denoted by $cr(G)$, is the minimum number of pairs of arcs (edges) that cross over all drawings of G.

A graph G is *planar* if and only if it has a drawing with no crossings.

Theorem 1.1 (Euler, 1750's). *Let $G = (V, E)$ be a connected planar graph drawn in the plan with $|F|$ faces. Then*

$$|V| - |E| + |F| = 2.$$

Note that G may contain multiple edges and loops.

Proof. Induction on $|V|$. Base case $|V| = 1$ is trivial. For $|V| > 2$, let $e = uv$ be an edge connecting two distinct vertices. By contracting e, we reduce the number of vertices and edges by one and the proof is complete by the induction hypothesis.

□

Corollary 1.2. *If G is a simple (no loops or multiple edges) planar graph with n vertices, then $|E(G)| \leq 3n - 6$.*

Proof. Let F be the set of faces in a planar drawing of G. Then

$$3|F| \leq \sum_{f \in F} |f| = 2|E|,$$

which implies $|F| \leq 2|E|/3$. Plugging this into Euler's formula gives

$$2 \leq |V| - |E| + (2|E|/3),$$

which implies $|E| \leq 3|V| - 6$.

$\square$

Corollary 1.3. *For any simple graph $G = (V, E)$, $cr(G) \geq |E| - 3|V|$.*

Proof. If $|E| > 3|V|$, and some drawing of G had fewer than $|E| - 3|V|$ crossings, then we can delete one edge from each crossing and obtain a planar graph with more than $3|V|$ edges. This contradiction completes the proof.

$\square$

Theorem 1.4 (Crossing Lemma, Ajtai-Chvá tal-Newborn-Szemerédi 1982, Leighton 1983)**.** *Let G be a simple graph. Then*

$$cr(G) \geq \frac{1}{64}\frac{|E|^3}{|V|^2} - |V|.$$

Proof. (Folklore) Consider any drawing of $G = (V, E)$ in the plane, with n vertices, m edges, and x crossings (pairs of edges that cross). We can assume that $m \geq 4n$, otherwise the statement is trivial. Choose a random subset $V' \subset V$ by picking each vertex $v \in V$ independently with probability p. Let $G' = (V', E')$ be the subgraph induced on V' with n' vertices, m' edges, and x' crossings in the *inherited* (same) drawing. Then $\mathbb{E}[n'] = np$, $\mathbb{E}[m'] = mp^2$, and $E[x'] = xp^4$. By Lemma 1.3 and linearity of expectation, we have

$$\mathbb{E}[x'] \geq \mathbb{E}[m'] - 3\mathbb{E}[n'],$$

which implies

$$xp^4 \geq mp^2 - 3np.$$

By setting $p = 4n/m$ (recall $m \geq 4n$), we have

$$x \geq \frac{1}{64}\frac{m^3}{n^2}.$$

$\square$

Proof of the Szemerédi-Trotter theorem. Consider a set P of m points and a set L of n lines in the plane maximizing $I(m, n)$. We define a drawing of a graph G where the vertices of G are the points from P, and two points p, q (vertices) are connected iff p and q lie on a common line $\ell \in L$ *next* to each other. Since a line with k points defines $k-1$ edges, our graph G contains $I(m, n) - n$ edges.

We have $cr(G) \leq \binom{n}{2} \leq n^2$, since two lines cross at most once. On the other hand, by the Crossing Lemma, we have

$$cr(G) \geq \frac{1}{64}\frac{|E|^3}{m^2} - m = \frac{1}{64}\frac{(I(m, n) - n)^3}{m^2} - m.$$

which implies $I(m, n) \leq O(m^{2/3}n^{2/3} + m + n)$.

Theorem 1.5 (Székely 1997)**.** *Let P be a set of m points and L be a set of n pseudo lines in the plane, that is, a family of curves in which every two members in L have at most one point in common. Then $I(P, L) \leq O(m^{2/3}n^{2/3} + m + n)$.*

References

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