Lecture 3

1 Point-line incidences

Let P be a set of m (distinct) points and L be a set of n (distinct) lines in the plane. An incidence is a pair $(p, \ell) \in P \times L$ such that $p \in \ell$. Denote $I(P, L) = \{(p, \ell) \in P \times L : p \in \ell\}$ and denote $I(m, n)$ to be the maximum $|I(P, L)|$ over all choices of an m-element point set P and an n-element set of lines L.

Theorem 1.1 (Szemerédi-Trotter, 1983)**.** *For all $m, n \geq 1$*

$$I(m, n) = O(m^{2/3}n^{2/3} + m + n)$$

and this bound is tight.

2 Incidences between points and curves

Theorem 2.1 (Szekely, 1997)**.** *Let P be a set of m points in the plane, and let C be a family of n pseudo-segments in the plane, that is, a family of curves in which any two members have at most 1 point in common. Then $|I(P, C)| \leq O(m^{2/3}n^{2/3} + m + n)$.*

What if we allow two curves in C to have two points in common? We say that a graph $G = (V, E)$ is a multi-graph with edge multiplicity k if any two vertices $u, v \in V(G)$ has at most k edges between them.

Lemma 2.2 (Crossing lemma for multi-graphs)**.** *Let $G = (V, E)$ be a multigraph with edge multiplicity k. Then*

$$cr(G) \geq \Omega\left(\frac{|E|^3}{k|V|^2}\right) - O(k^2|V|).$$

Proof. draw G in the plane with $cr(G)$ crossings. Consider each edge independently, and delete it with probability $1 - 1/k$. After all edges are considered, delete all edges between uv if it is a multi-edge (at least 2). Let $G' = (V, E')$ be the resulting graph (with the inherited drawing). Let p_e denote the probability that edge e survives. Then clearly $p_e \leq 1/k$. On the other hand, e survives if it is not deleted, and all the other $k' \leq k$ edges are deleted. Hence

$$p_e \geq \frac{1}{k}\left(1 - \frac{1}{k}\right)^{k'-1} \geq \frac{1}{3k}.$$

By the crossing lemma (G' is simple), we have

$$cr(G') - \frac{|E'|^3}{64|V|^2} + |V| \geq 0.$$

Therefore

$$\mathbb{E}[cr(G')] \geq \frac{\mathbb{E}[|E'|^3]}{64|V|^2} - |V|.$$

By Jensen's inequality, we get $\mathbb{E}[|E'|^3] \geq (\mathbb{E}[E'])^3$. Also since $p_e \leq 1/k$, we have $\mathbb{E}[cr(G')] \leq cr(G)/k^2$. Putting this together gives

$$\frac{cr(G)}{k^2} \geq \Omega\left(\frac{|E|^3}{k^3|V|^2}\right) - O(|V|).$$

□

3 Application of Szemeredi-Trotter: Sums versus product

Theorem 3.1 (Elekes, 1997)**.** *Let $A = \{a_1, \ldots, a_n\}$ be a set of n real numbers, and define $A + A = \{a + b : a, b \in A\}$ and $AA = \{ab : a, b \in A\}$. Then we have*

$$\max\{|A + A|, |AA|\} \geq \Omega(n^{5/4}).$$

Proof. Define the point set $P = \{(x, y) \in \mathbb{R}^2 : x \in A + A, y \in AA\}$. We define a set of n^2 lines L by the functions

$$f_{i,j} = a_i(x - a_j).$$

for $i, j \in \{1, \ldots, n\}$. For each $\ell = 1, \ldots, n$, the line $f_{i,j}$ is incident to the point $(a_j + a_\ell, a_i a_\ell) \in P$. Therefore each line in L is n-rich, and by the Szemerédi-Trotter theorem we have

$$n^2 = |L| \leq O\left(\frac{|P|^2}{n^3} + \frac{|P|}{n}\right) \leq \frac{(|AA||A + A|)^2}{n^3}$$

□

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