Lecture 4

1 Point-line incidences

Let P be a set of m (distinct) points and L be a set of n (distinct) lines in the plane. An incidence is a pair $(p,\ell) \in P \times L$ such that $p \in \ell$. Denote $I(P,L) = \{(p,\ell) \in P \times L : p \in \ell\}$ and denote $I(m,n)$ to be the maximum $|I(P,L)|$ over all choices of an m-element point set P and an n-element set of lines L.

Theorem 1.1 (Szemerédi-Trotter, 1983). *For all $m,n \geq 1$*

$$I(m,n) = O(m^{2/3}n^{2/3} + m + n)$$

and this bound is tight.

2 Cell decomposition

Theorem 2.1 (Cutting Lemma). *Let L be a set of n lines in the plane, and let r be a parameter, $1 < r < n$. Then the plane can be subdivided into t generalized triangles (this means intersections of three half-planes) $\Delta_1, \Delta_2, ...\Delta_t$ in such a way that the interior of each Δ_i is intersected by at most n/r lines of L, and we have $t = O(r^2)$.*

Theorem 2.2 (Weak Cutting Lemma). *Let L be a set of n lines in the plane, and let r be a parameter, $1 < r < n$. Then the plane can be subdivided into t generalized triangles (this means intersections of three half-planes) $\Delta_1, \Delta_2, ...\Delta_t$ in such a way that the interior of each Δ_i is intersected by at most n/r lines of L, and we have $t = O(r^2 \log^2 n)$.*

Proof. Set $s = 6r \ln n$, and select a random sample $S \subset L$ of the given lines, by making s independent random draws, drawing a random line from L each time with replacement (i.e one line can be selected several times). Consider the arrangement of S. Any cell that is not a "triangle", partition it further by adding diagonals connecting its vertices (say pick one vertex and draw segments from it to all other vertices within the cell). We will show that there is a sample S that gives rise to the collection of triangles as desired.

We say a triangle $T \subset \mathbb{R}^2$ is **dangerous** if its interior is intersected by at least n/r lines from L. Fix an arbitrary dangerous triangle T. Then the probability no line in our sample S intersect the interior of T is at most $(1 - 1/r)^s$. Using the well-known inequality $1 + x \leq e^x$ we have this is at most $e^{-6 \ln n} = n^{-6}$.

Now call a triangle T **interesting** if it can appear in a triangulation for some sample $S \subset L$. Notice that all interesting triangles has vertices at some three vertices of the arrangement of L. Therefore there are only at most n^6 possible interesting triangles. Therefore with positive probability, a random sample will intersect the interiors (kill off) all dangerous-interesting triangles simultaneously. Therefore all triangles in our sample has the property that a most n/r lines intersect its interior and we are done.

$\square$

P is a set of m points and L is a set of n lines. Since the incidence graph is $K_{2,2}$-free, we have the weak bound (using Kovari-Sos-Turan)

$$|I(m,n)| \leq O(n\sqrt{m} + m)$$

and

$$|I(m,n)| \leq O(m\sqrt{n} + n)$$

However these bounds are good if the problem is off-balanced.

Another Proof of Szemeredi-Trotter. Here we show again n points n lines have at most $O(n^{4/3})$ incidences. We apply the cutting lemma with parameter $r = n^{1/3}$, and divide the plane into at most $t = O(r^2) = O(n^{2/3})$ generalized triangles $\Delta_1, ..., \Delta_t$. We partition our points into two parts $P = V \cup P_0$, where V are points that lie on the vertex of some Δ_i, P_0 are the remaining points (not at the vertex of Δ_i). Notice that $|V| \leq 3t = O(n^{2/3})$. Hence using the weak bound we have $|I(V,L)| \leq O(n^{2/3}\sqrt{n} + n) < O(n^{4/3})$.

Let P_i denote the set of points from P that lies inside Δ_i, but not on the vertices of Δ_i. Let L_i be the set of lines intersecting the interior of Δ_i. By the cutting lemma, $|L_i| \leq n/r = O(n^{2/3})$.

$$\sum_{i=1}^{t} |I(L_i, P_i)| \leq \sum_{i=1}^{t} I(n^{2/3}, |P_i|) = \sum_{i=1}^{t} O(|P_i|n^{1/3} + n^{2/3}) = O(n^{4/3}).$$

Finally, the only incidence pairs we have not counted are points on the boundary of Δ_i, but not at the vertex, and lines that borders Δ_i. Let L_b denote all lines that borders Δ_i for some i. Then $|L_b| \leq O(n^{2/3})$. Just as before, $|I(P, L_b)| \leq O(n^{2/3}\sqrt{n} + n) < O(n^{4/3})$.

References

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