Lecture 3

1 Simplicial partitioning via cuttings

Given a set of points $P_i \subset \mathbb{R}^2$, a line ℓ crosses P_i if not all members of P_i lie in one of the two closed half-planes defined by ℓ.

Theorem 1.1. *For any integer $r \geq \log^2 n$, every n element planar point set P can be partitioned into at r part $P = P_1 \cup \cdots \cup P_r$ such that every line ℓ intersects the interior of at most $O(\sqrt{r})$ triangles Δ_i.*

Proof. It is enough to prove the statement for the lines spanned by the point set P. Let L be the set of at most n^2 lines spanned by the pairs of points in P. We apply the cutting lemma to L with parameter $c\sqrt{r}$, and subdivide $\mathbb{R}^2 = \Delta_1 \cup \cdots \cup \Delta_r$. By the pigeonhole principle, there is a part Δ that contains at least n/r points from P. Set P_1 to be a set of exactly n/r points inside Δ. For each line ℓ that crosses Δ, we double it. By the cutting lemma, at most $n^2/c\sqrt{r}$ lines crosses Δ_1. Let L_1 be the set of all lines now considered, that is $|L_1| \leq n^2 + n^2/c\sqrt{r}$.

We repeat the argument on the point set $P \setminus P_1$ and the set of lines L_1. We apply the cutting lemma to L_1 with parameter $c\sqrt{r-1}$ and subdivide $\mathbb{R}^d = \Delta_1 \cup \cdots \cup \Delta_{r-1}$. By the pigeonhole principle, there is a Δ that contains at least $n(1-1/r)/(r-1) = n/r$ points from P. Set P_2 be the set of exactly n/r points inside Δ. We again double all lines intersecting the interior of Δ, and by the cutting lemma we doubled at most $\frac{|L_1|}{c\sqrt{r-1}}$ lines from L_1. Set L_2 to be the set of all current lines, which implies

$$L_2 \leq n^2\left(1+\frac{1}{c\sqrt{r}}\right) + \frac{n^2\left(1+\frac{1}{c\sqrt{r}}\right)}{c\sqrt{r-1}} = n^2\left(1+\frac{1}{c\sqrt{r}}\right)\left(1+\frac{1}{c\sqrt{r-1}}\right)$$

We continue this process, such that at the i-th step, we obtain $P_1, \ldots, P_i$, and a set of lines L_i, and apply the argument to $P \setminus (P_1 \cup \cdots \cup P_i)$, which consists of $n - in/r = n(1-i/r)$ points, and

$$L_i \leq n^2\left(1+\frac{1}{c\sqrt{r}}\right)\left(1+\frac{1}{c\sqrt{r-1}}\right)\cdots\left(1+\frac{1}{c\sqrt{r-i-1}}\right)$$

In the end we obtain our partition $P_1, \ldots, P_r$ and triangles $\Delta_1, \ldots, \Delta_r$. Moreover, we will have a family of lines L_r such that

$$|L_r| \leq n^2 e^{6\sum_{i=1}^{r}\frac{1}{c\sqrt{i}}} \leq n^2 e^{O(\sqrt{r})}.$$

Suppose line ℓ intersects κ triangles Δ_i. Then the number of copies of ℓ in L_r is 2^κ. Combing the above inequality gives $\kappa \leq O(\sqrt{r})$, for $r \geq \log^2 n$. $\square$

Corollary 1.2. *Given n points in the plane in general position, where n is even, there is a matching of size $n/2$ such that any line intersects at most $O(\sqrt{n})$ members in the matching.*

Proof. Apply the theorem with parameter $r = n/2$.

□

A geometric graph G is a graph drawn in the plane with vertices as points, and edges as straight line segments. Two edges $e_1, e_2 \in E(G)$ are *mutually avoiding* if they form the opposite sides of a convex 4-gon (in other words, the line through one edge does not stab the other).

Theorem 1.3. *If G is a complete n vertex geometric graph, then G contains $\sqrt{n}$ pairwise mutually avoiding edges.*

Proof. Let M be the matching obtained from the corollary above. Let $G' = (M, E)$ be the graph whose vertices are the members in M, and two vertices are adjacent if and only if they are not mutually avoiding. By the corollary, the out degree of each vertex is at most $O(\sqrt{n})$, which implies that $|E(G')| \le O(n^{3/2})$. By Turan's theorem, G contains an independent set of size $\Omega(\sqrt{n})$. The corresponding edges are pairwise mutually avoiding.

□

References

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