Lecture 6

1 Lower bound construction

Let

$$\mathcal{P} = \left\{ (i,j) | 1 \le i \le \sqrt{n} \quad \text{and} \quad -\frac{\sqrt{n}}{2} \le j \le \frac{\sqrt{n}}{2} \right\}.$$

Given a pair of relatively prime integers $a < b$, we define the line set:

$$\mathcal{L}_{a,b} = \left\{ y - j = \frac{a}{b}(x-i) | \quad 1 \le i \le b, \quad 1 \le j \le \frac{\sqrt{n}}{4} \right\}.$$

Notice that $|\mathcal{L}_{a,b}| = b\sqrt{n}/4$. Our set of lines is the union of many sets of the form $\mathcal{L}_{a,b}$. Specifically, we set $k = c' n^{1/6}$ (for a constant c' that will be set below) and

$$\mathcal{L} = \bigcup_{\substack{1 \le a < b \le k \\ gcd(a,b)=1}} \mathcal{L}_{a,b}.$$

We claim that the lines of $\mathcal{L}$ are distinct. Indeed, two lines from the same set $\mathcal{L}_{a,b}$ are distinct since they intersect the x-axis in distinct points. Two lines from two different sets are distinct since they have distinct slopes. We recall Eulers totient function:

$$\phi(b) = |\{ \ a \mid 1 \le a < b \quad \text{and} \quad \gcd(a,b) = 1 \ \}| .$$

Notice that

$$|\mathcal{L}| = \sum_{1 \le b \le k} \left(\phi(b) \cdot \frac{b\sqrt{n}}{4} \right).$$

Since $\sum_{i=1}^{n} i \cdot \phi(i) = \Theta(n^3)$ (e.g., this is a variant of the proof of Theorem 330 in this book by Hardy and Wright), we have

$$|\mathcal{L}| = \frac{\sqrt{n}}{4} \sum_{1 \le b \le k} b \cdot \phi(b) = \Theta\left(\sqrt{n}k^3\right) = \Theta\left((c')^3 n\right).$$

Thus, by an appropriate choice of c' , we obtain $|\mathcal{L}| = n$. It remains to bound the number of incidences in $\mathcal{P} \times \mathcal{L}$. Consider a line $\ell \in \mathcal{L}$ that is defined by $y - j = \frac{a}{b}(x-i)$ and notice that ℓ is incident to a point in every r-th column of the integer lattice. The slope of ℓ is smaller than one and it intersects the y-axis below the point $(0, \sqrt{n}/4)$. Moreover, by setting c to be sufficiently large (with respect to c') so that $cn^{1/6} < \sqrt{n}/2$, we obtain that ℓ intersects the y-axis above the point $(0, -\sqrt{n}/2)$. Combining these properties, we notice that ℓ is incident to at least $\frac{\sqrt{n}}{4b} - 1$ points of $\mathcal{P}$, and thus

$$|I(\mathcal{P},\mathcal{L})| \geq \sum_{\substack{1\leq a<b\leq k \\ gcd(a,b)=1}} |\mathcal{L}_{a,b}| \cdot \left(\frac{\sqrt{n}}{4b} - 1\right) = \sum_{1\leq b\leq k} \phi(b)\frac{b\sqrt{n}}{4}\left(\frac{\sqrt{n}}{4b} - 1\right) = \Omega\left(n \sum_{1\leq b\leq k} \phi(b)\right).$$

Since $\sum_{i=1}^{n} \phi(i) = \Theta(n^2)$ (e.g., see Theorem 330 in this book by Hardy and Wright), we have

$$I(\mathcal{P},\mathcal{L}) = \Omega\left(mb^2\right) = \Omega(m^{2/3}n^{2/3}).$$