Semi-algebraic Ramsey numbers and its applications

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Introduction

For k-uniform hypergraphs.

Definition

We define the *Ramsey number* $R_k(n)$ to be the minimum integer N such that any N-vertex k-uniform hypergraph H contains either a clique or an independent set of size n.

Theorem (Ramsey 1930)

For all k, n, the Ramsey number $R_k(n)$ is finite.

Estimate $R_k(n)$, k fixed and $n \to \infty$.

Known estimates

Theorem (Erdős-Szekeres 1935, Erdős 1947)

$$2^{n/2} \leq R_2(n) \leq 2^{2n}.$$

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Theorem (Erdős-Rado 1952)

$$2^{cn^2} \leq R_3(n) \leq 2^{2^{c'n}}.$$

Known estimates

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Theorem (Erdős-Rado 1952, Erdős-Hajnal 1960's)

$$2^{cn^2} \leq R_3(n) \leq 2^{2^{c'n}}.$$

$$t_{k-1}(cn^2) \leq R_k(n) \leq t_k(c'n).$$

Tower function $t_i(x)$ is given by $t_1(x) = x$ and $t_{i+1}(x) = 2^{t_i(x)}$.

Combinatorial Problem

$$2^{cn^2} \leq R_3(n) \leq 2^{2^{c'n}}.$$

Conjecture (Erdős, \$500 problem)

$$2^{2^{cn}} \leq R_3(n)$$

Erdős-Hajnal Stepping Up Lemma: $x < R_k(n)$, then $2^x \lesssim R_{k+1}(n)$ for $k \geq 3$

Would imply $R_4(n) = 2^{2^{2^{\Theta(n)}}}$, and $R_k(n) = t_k(\Theta(n))$.

Combinatorial Problem

$$2^{cn^2} \leq R_3(n) \leq 2^{2^{c'n}}.$$

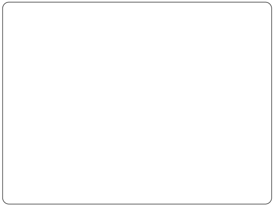
Conjecture (???)

$$R_3(n) \leq 2^{cn^2}$$

Erdős-Rado: $R_k(n) \leq 2^{(R_{k-1}(n))^c}$

$R_3(n) \leq 2^{(R_2(n))^2}$.

H



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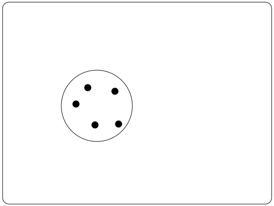

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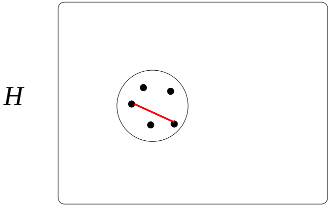

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Betters bounds on Ramsey problems in discrete geometry.

Higher order Erdős-Szekeres: Find *Order-type homogeneous* subsequence.

Given a point sequence $P = p_1, p_2, \ldots, p_N \subset \mathbb{R}^d$ in general position, $\chi : \binom{P}{d+1} \rightarrow \{+1, -1\}$ (positive or negative orientation).

$$\chi(p_1, \ldots, p_{d+1}) = \mathrm{sgn}\,\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ | & | & \cdots & | \\ p_{i_1} & p_{i_2} & \vdots & p_{i_{d+1}} \\ | & | & \cdots & | \end{pmatrix}.$$

χ is the *order-type* of P.

A point sequence $P = p_1, ..., p_n \subset \mathbb{R}^d$ is *order-type homogeneous*, if every $d+1$-tuple has the same orientation (i.e. all positive or all negative).

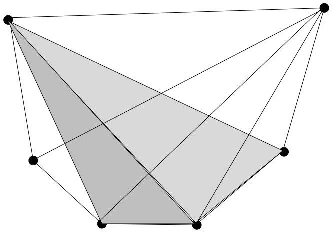
Fact

A point sequence that is order-type homogeneous forms the vertex set of a convex polytope combinatorially equivalent to the cyclic polytope in $\mathbb{R}^d$.

A point sequence $P = p_1, ..., p_n \subset \mathbb{R}^d$ is *order-type homogeneous*, if every $d + 1$-tuple has the same orientation (i.e. all positive or all negative).

Theorem (McMullen 1962)

Among all d-dimensional convex polytopes with n vertices, the cyclic polytope maximizes the number of faces of each dimension



A point sequence $P = p_1, ..., p_n \subset \mathbb{R}^d$ is *order-type homogeneous*, if every $d+1$-tuple has the same orientation (i.e. all positive or all negative).

Problem (Corodovil-Duchet 2000, Matoušek-Eliáš 2012.)

Determine the minimum integer $OT_d(n)$, such that any sequence of $OT_d(n)$ points in $\mathbb{R}^d$ in general position, contains an n-element subsequence that is order-type homogeneous.

$$OT_d(n) \leq R_{d+1}(n).$$

1-dimension: $P = p_1, ..., p_N \subset \mathbb{R}$, order-type homogeneous subset

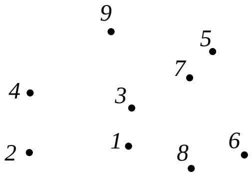
$$p_{i_1} < p_{i_2} < \cdots < p_{i_n}$$

or

$$p_{i_1} > p_{i_2} > \cdots > p_{i_n}$$

Erdős-Szkeres (1935): $OT_1(n) = (n-1)^2 + 1$

2-dimensions: Order-type homogeneous subset: Every triple has a clockwise orientation, or every triple has a clockwise orientation.



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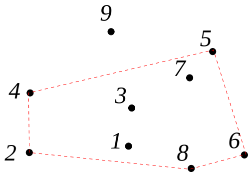
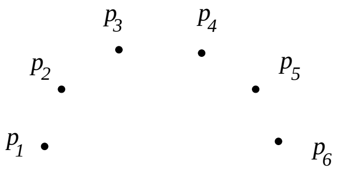


2-dimensions: Order-type homogeneous subset: Every triple has a clockwise orientation, or every triple has a clockwise orientation.

$OT_2(n)$ is about points in convex position (Happy Ending Problem).



Erdős-Szkeres cups-caps Theorem (1935): $OT_2(n) = 2^{\Theta(n)}$

$OT_1(n) = \Theta(n^2), OT_2(n) = 2^{\Theta(n)}$. (Based on transitivity)

For $d \geq 3$

$V = \{N$ labeled points in $\mathbb{R}^d$ in general position$\}$
$E = \{(d+1)$-tuples having a positive orientation$\}$

- $OT_d(n) \leq R_{d+1}(n) \leq t_{d+1}(O(n))$

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ | & | & \cdots & | \\ p_{i_1} & p_{i_2} & \vdots & p_{i_{d+1}} \\ | & | & \cdots & | \end{pmatrix} > 0.$$

- **Conlon, Fox, Pach, Sudakov, S. 2012**: $OT_d(n) \leq t_d(n^{c_d})$, where c_d is exponential in a power of d.

Theorem (S. 2013)

For $d \geq 2$, we have

$$OT_d(n) \leq t_d(O(n))$$

Lower bound: $OT_3(n) \geq 2^{2^{\Omega(n)}}$ (Elias-Matousek 2012).
$OT_d(n) \geq t_d(\Omega(n)), d \geq 4$ (Barany-Matousek-Por 2014)

- $OT_1(n) = \Theta(n^2)$ (Erdős-Szekeres 1935)
- $OT_2(n) = 2^{\Theta(n)}$ (Erdős-Szekeres 1935/1960)
- $OT_3(n) = 2^{2^{\Theta(n)}}$ (Elias-Matousek 2012, S. 2013)
- $OT_d(n) = \mathrm{twr}_d(\Theta(n))$ (Barany-Matousek-Por 2014, S. 2013)

Tight bounds in all dimensions!

- $OT_1(n) = \Theta(n^2)$ (Erdős-Szekeres 1935)
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V = points in $\mathbb{R}^d$

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ | & | & \cdots & | \\ p_{i_1} & p_{i_2} & \vdots & p_{i_{d+1}} \\ | & | & \cdots & | \end{pmatrix} > 0.$$

- $OT_1(n) = \Theta(n^2)$ (Erdős-Szekeres 1935)
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Generalization: Semi-algebraic hypergraphs.

We say that $H = (V, E)$ is a **semi-algebraic k-uniform hypergraph in d-space** if

$V = \{n \text{ points in } \mathbb{R}^d\}$

E defined by polynomials $f_1, ..., f_t$ and a Boolean formula Φ such that

$$(p_{i_1}, ..., p_{i_k}) \in E$$

$$\Leftrightarrow \Phi(f_1(p_{i_1}, ..., p_{i_k}) \geq 0, ..., f_t(p_{i_1}, ..., p_{i_k}) \geq 0) = \text{yes}$$

Think of t as a constant.

Definition

We define the *Ramsey number* $R_k^{semi}(n)$ to be the minimum integer N such that any N-vertex k-uniform **semi-algebraic** hypergraph H (in $\mathbb{R}^d$) contains either a clique or an independent set of size n.

$R_k^{semi}(n) \leq R_k(n)$.

Theorem (Alon, Pach, Pinchasi, Radoičić, Sharir 2005)

$$R_2^{semi}(n) \leq n^{c_1}.$$

$R_k^{semi}(n) \leq 2^{R_{k-1}^{semi}(n)}$.

Theorem (Conlon, Fox, Pach, Sudakov, S. 2012)

for $k \geq 3$,

$$t_{k-1}(c_2 n) \leq R_k^{semi}(n) \leq t_{k-1}(n^{c_1}).$$

Recall: for $k \geq 3$, $t_{k-1}(cn^2) \leq R_k(n) \leq t_k(c'n)$.

Theorem (Conlon, Fox, Pach, Sudakov, S. 2012)

for $k \geq 3$,

$$R_k^{semi}(n) \leq t_{k-1}(n^{c_1}).$$

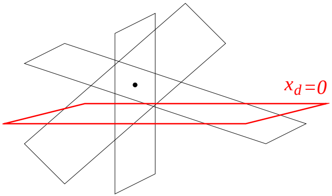
Several applications...

Problem (Matoušek-Welzl 1992, Dujmović-Langerman 2011, Matoušek-Eliáš 2012.)

Determine the minimum integer $OSH_d(n)$, such that any family of at least $OSH_d(n)$ hyperplanes in $\mathbb{R}^d$ in general position, must contain n members such that every d-tuple intersects on one-side of the hyperplane $x_d = 0$.

$$OSH_2(n) = \Theta(n^2), \qquad OSH_d(n) \leq R_d(n) \leq t_d(c'n).$$

$V = \{\text{N hyperplanes}\}$,
$E = \{d\text{-tuples that intersect above } x_d = 0 \text{ hyperplane}\}$.
New bound: $OSH_d(n) \leq R_d^{semi}(n) \leq t_{d-1}(n^{c_1})$



Combinatorial Problem

Ramsey number of 3-uniform hypergraphs.

$$2^{cn^2} \leq R_3(n) \leq 2^{2^{c'n}}.$$

Conjecture (Erdős)

$$2^{2^{cn}} \leq R_3(n)$$

Is there a geometric construction showing $2^{2^{cn}} \leq R_3(n)$?

Our Result: $R_3^{semi}(n) \leq 2^{n^{c_1}}$.

Geometric Ramsey Problems.

$OT_d(n) = \mathrm{twr}_d(\Theta(n))$

$R_k^{semi}(n) = \mathrm{twr}_{k-1}(n^{\Theta(1)})$.

Classical Ramsey Numbers, $k \geq 3$

$$R_2(n) = 2^{\Theta(n)}$$

$$\mathrm{twr}_{k-1}(cn^2) \leq R_k(n) \leq \mathrm{twr}_k(c'n)$$

Off diagonal Ramsey numbers

$R_k(s,n)$, clique of size s, independent set of size n. s fixed, $n \to \infty$.

Graphs:
$R_2(3,n) = \Theta(n^2 \log n)$ (Ajtai-Komlós-Szemerédi 1980, Kim 1995)
$R_2(s,n) = n^{\Theta(1)}$

Theorem (Spencer 1978, Ajtai, Komlós, Szemerédi 1980)

For fixed s

$$n^{s/2} \leq R_2(s,n) \leq n^s$$

Off diagonal Ramsey numbers

Graphs: $R_2(s,n) \leq O(n^s)$

Recursive formula: $R_3(s,n) \leq 2^{(R_2(s,n))^2}$

Theorem (Erdős-Hajnal-Rado 1952, 1965)

Fixed s

$$2^n \leq R_3(s,n) \leq 2^{n^{2s}}.$$

$$\mathrm{twr}_{k-1}(n) \leq R_k(s,n) \leq \mathrm{twr}_{k-1}(n^{2s}).$$

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Theorem (Conlon-Fox-Sudakov 2011)

Fixed s

$$2^{n\log n} \leq R_3(s,n) \leq 2^{n^s}.$$

$$\mathrm{twr}_{k-1}(n\log n) \leq R_k(s,n) \leq \mathrm{twr}_{k-1}(n^s).$$

Off diagonal Ramsey numbers for Semi-algebraic hypergraphs

Graphs: (Walczak 2014, Conlon-Fox-Pach-Sudakov-S. 2013)

$$n \log\log n \leq R_2^{semi}(s,n) \leq n^C.$$

$C = 2^d$

3-uniform hypergraphs: (Conlon-Fox-Pach-Sudakov-S. 2013)

$$n^c \leq R_3^{semi}(s,n) \leq 2^{n^{c'}}.$$

$$\mathrm{twr}_{k-2}(n^c) \leq R_k^{semi}(s,n) \leq \mathrm{twr}_{k-1}(n^{c'}).$$

$$n^c \leq R_3^{semi}(s,n) \leq 2^{n^{c'}}.$$

Conjecture (Conlon-Fox-Pach-Sudakov-S. 2014)

For fixed s,

$$R_3^{semi}(s,n) \leq n^C$$

$R_3^{semi} \leq 2^{R_2^{semi}(s,n)}$

H

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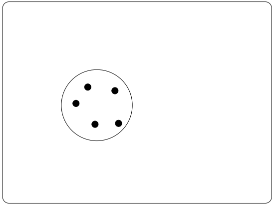

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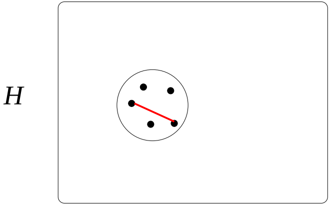


$$\Omega(n^c) \leq R_3^{semi}(s,n) \leq 2^{n^C}.$$

Conjecture (Conlon-Fox-Pach-Sudakov-S. 2014)

For fixed s,

$$R_3^{semi}(s,n) \leq n^C$$

Difficulties

- $R_3^{semi} \leq 2^{R_2^{semi}(s,n)}$.

Using different methods (Not Erdős-Rado)

Theorem (S. 2014+)

For fixed k, s

$$R_3^{semi}(s, n) \leq 2^{n^{o(1)}}.$$

$R_k^{semi}(s, n) \leq 2^{R_{k-1}^{semi}(s,n)}$

Corollary

For fixed k, s

$$R_k^{semi}(s, n) \leq \mathrm{twr}_k(n^{o(1)}).$$

Application

Problem

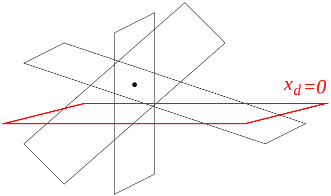
Off diagonal One-Sided Hyperplane problem: $OSH_d(s,n)$

$$OSH_2(s,n) = O(sn)$$

$OSH_d(s,n) \leq 2^{OSH_{d-1}(s,n)}$

Conlon-Fox-Pach-Sudakov-S.: $OSH_3(s,n) \leq 2^{O(sn)}$

S., 2014+ $OSH_3(s,n) \leq 2^{n^{o(1)}}$.



$R_3^{semi}(s,n) \leq 2^{n^{o(1)}}$

Theorem (S. 2014+)

Let $H = (V, E)$ be a semi-algebraic 3-hypergraph on N vertices. If H is $K_4^{(3)}$-free, then H has an independent set S of size

$$|S| \geq 2^{\frac{(\log\log N)^2}{c\log\log\log N}}$$

Sketch Proof: 3-uniform hypergraph $H = (V, E)$, $V = \{p_1, ..., p_N\}$ points in $\mathbb{R}^d$.

Relation $E \subset \binom{V}{3}$ depends on f and Φ

$$\phi(f(x_1, x_2, x_3) \geq 0) = \{\text{yes,no}\}$$

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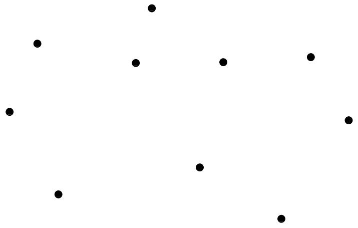
$(p_1, p_2, p_3) \in E$ depends on $f(p_1, p_2, p_3) \rightarrow \{+, -, 0\}$.



3-uniform hypergraph $H = (V, E)$, $V = \{p_1, ..., p_n\}$ points in $\mathbb{R}^d$.

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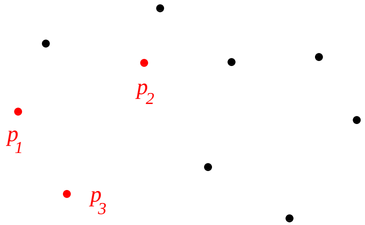


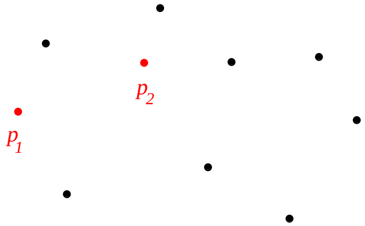
Example

3-uniform hypergraph $H = (V, E)$, $V = \{p_1, ..., p_n\}$ points in $\mathbb{R}^d$.

Relation $E \subset \binom{V}{3}$ depends on f and Φ

$$\phi(f(x_1, x_2, x_3) \geq 0) = \{\text{yes,no}\}$$

Zero set $f(p_1, p_2, x_3) = 0$, surface in $\mathbb{R}^d$.



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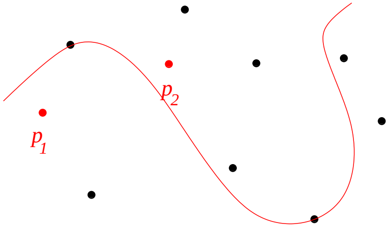
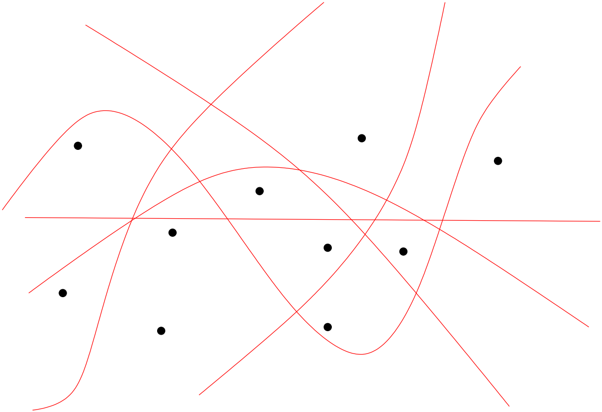
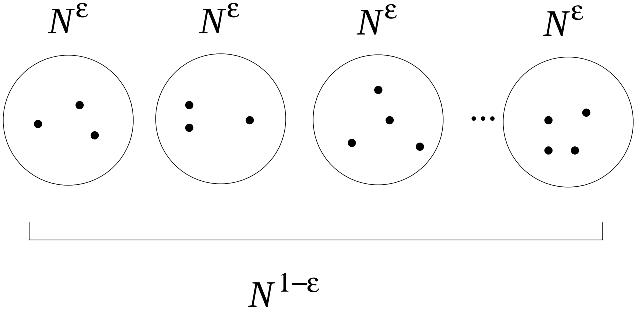


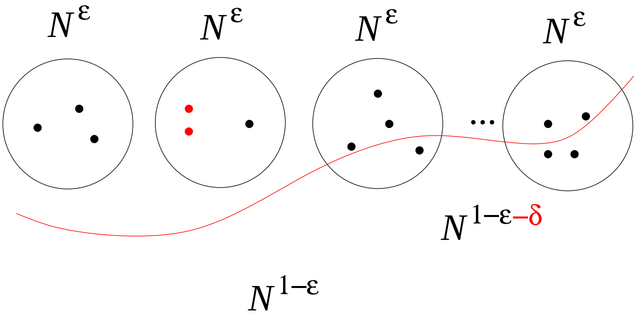
$\binom{N}{2}$ surfaces in $\mathbb{R}^d$ and N points.



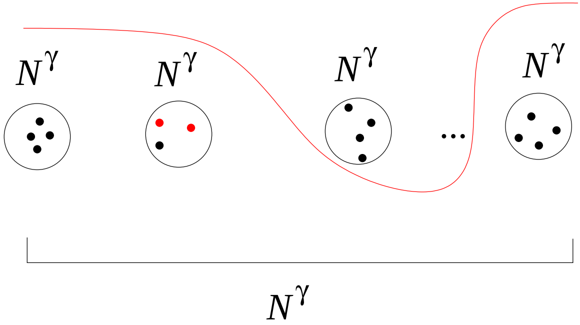
Using cell decomposition. $\epsilon = \epsilon(d)$



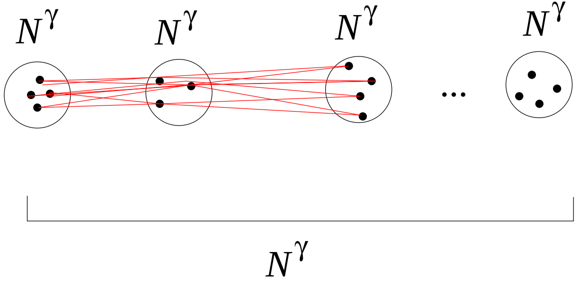
Using cell decomposition. $\epsilon = \epsilon(d)$, $\delta = \delta(d)$



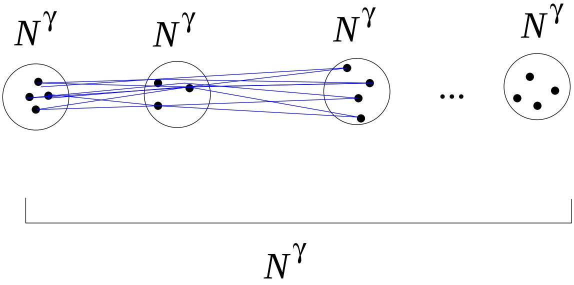
$\gamma = \gamma(d)$.



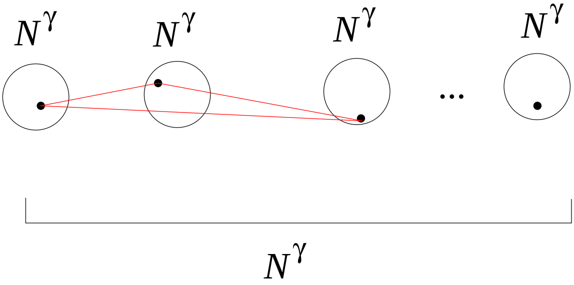
By a probabilistic argument. Red represents edges.



By a probabilistic argument. Blue represents NON-edges.



Apply induction on the N^{γ} vertices



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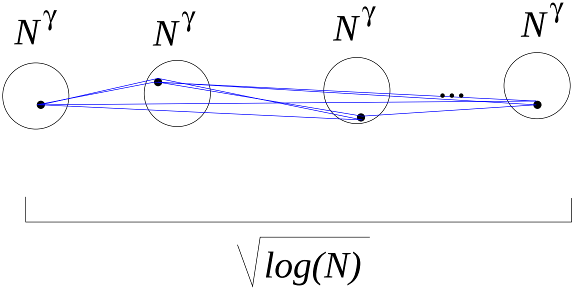
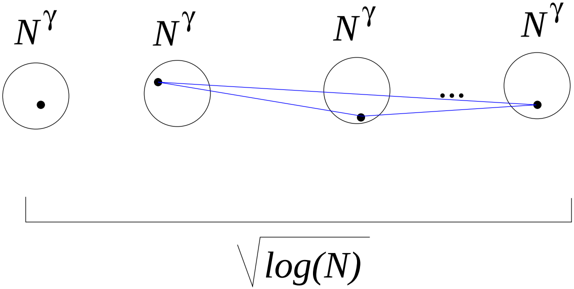
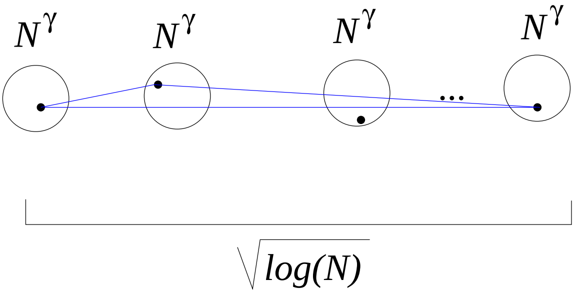


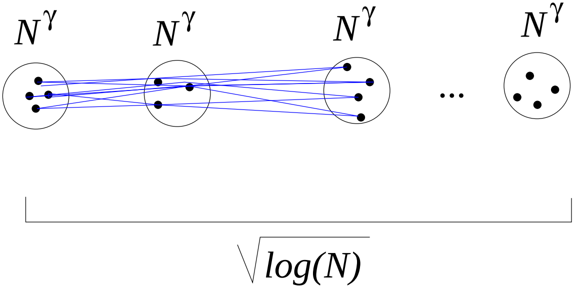
Apply induction on the N^{γ} vertices



Apply induction on the N^γ vertices



Apply induction on the N^γ vertices



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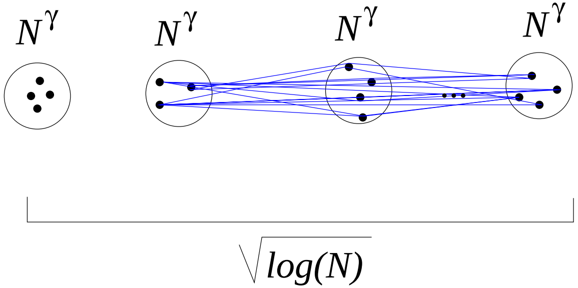
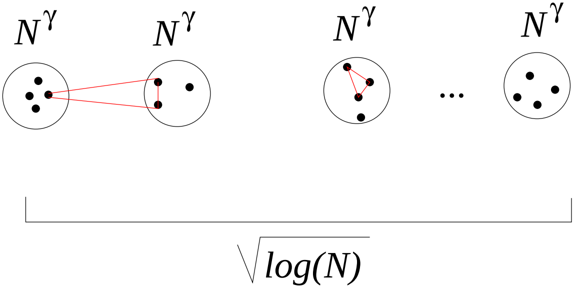
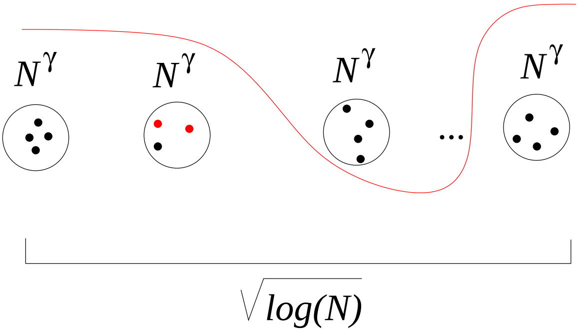


Apply induction on the N^γ vertices



(2, 1) situation



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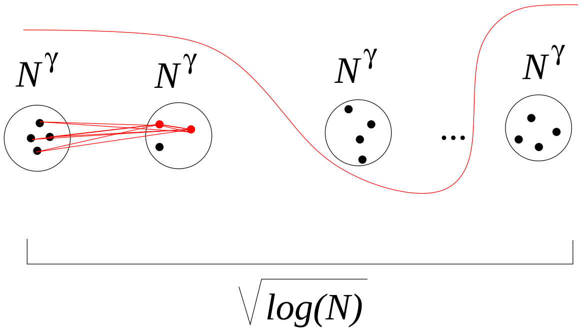

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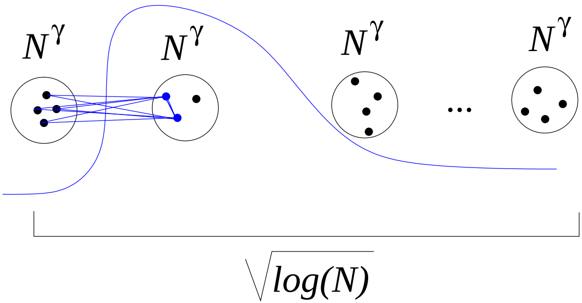
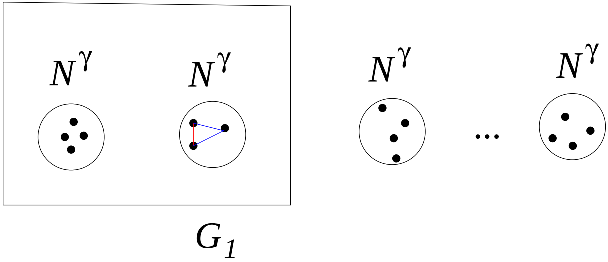
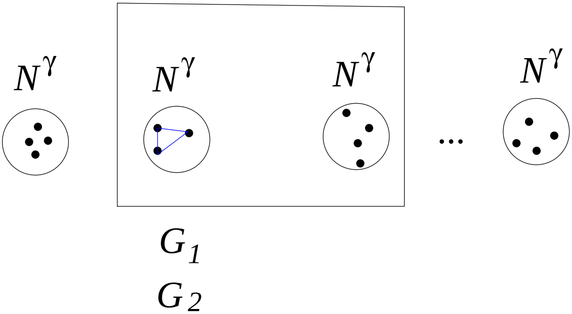


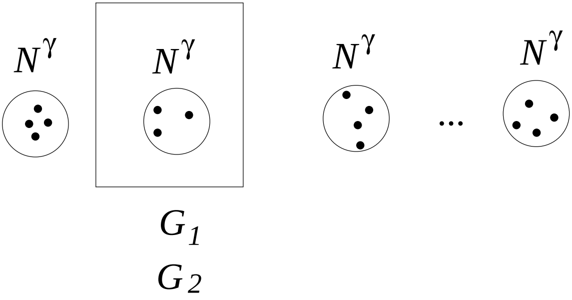
Graph G_1



Graph G_2



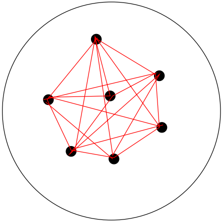
Collection of graphs.

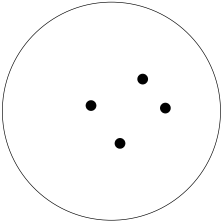


Graphs $G_1, G_2, \ldots, G_{\sqrt{\log N}}$ are each semi-algebraic.

Semi-algebraic graphs $G_1, G_2, \ldots, G_{\sqrt{logN}}$ on vertex set V', $|V'| = N^{\gamma}$,

Case 1: $\exists G_i$ with clique of size $2^{(\log N)^{1/4}} > 2^{(\log\log N)^2/\log\log\log N}$.





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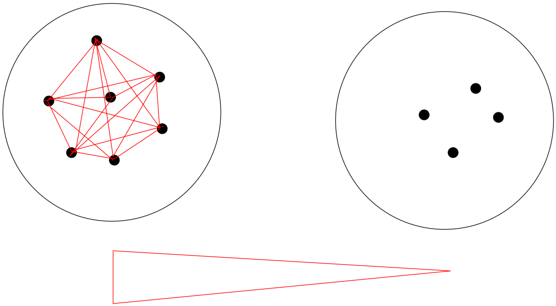

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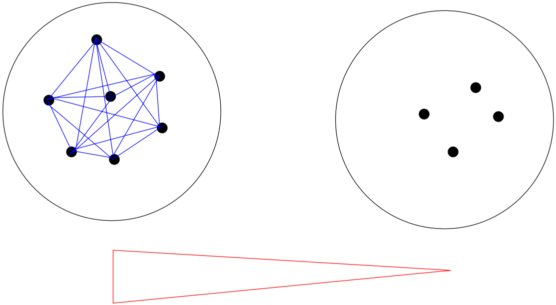
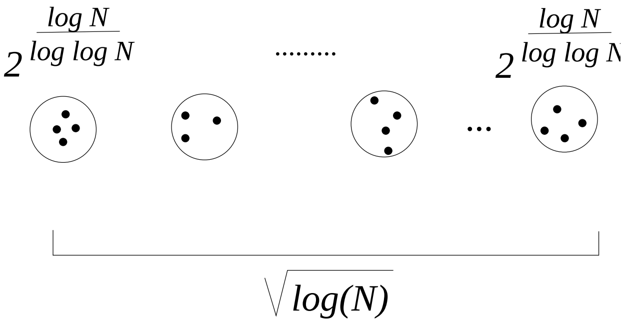


Recall H is $K_4^{(3)}$-free! Assume $\omega(G_i) \leq 2^{(\log N)^{1/4}}$.

Lemma

Semi-algebraic graphs $G_1, \ldots, G_{\sqrt{\log N}}$ *with* $\omega(G_i) \leq 2^{(\log N)^{1/4}}$ *on* V', $|V'| = N^{\gamma}$. *Then* $\exists S \subset V'$ *such that* $|S| = 2^{(\log N)/\log\log N}$ *such that* $G_i[S]$ *is empty for all* i



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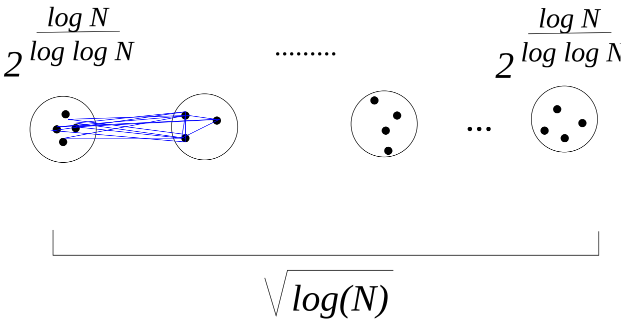

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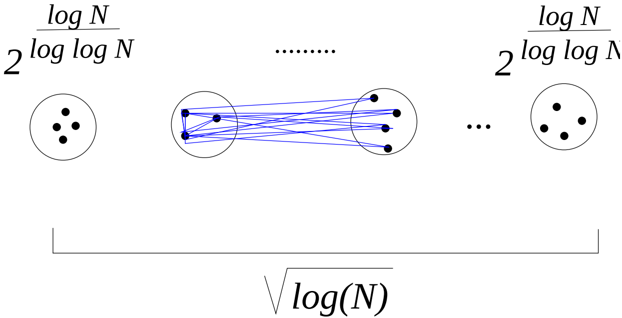

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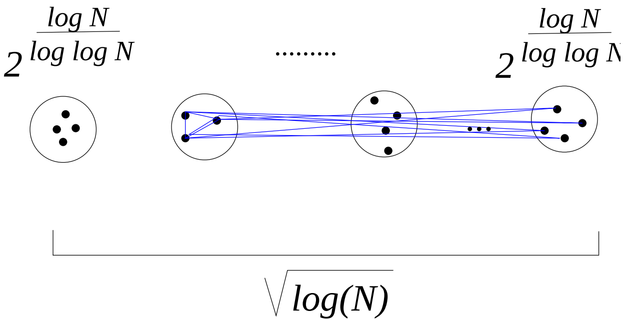

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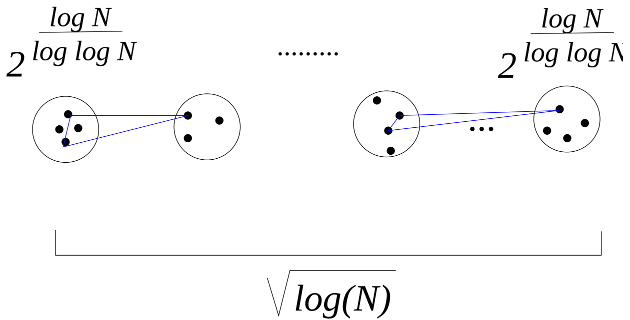

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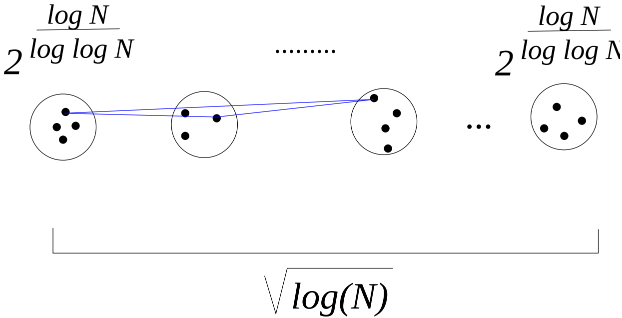

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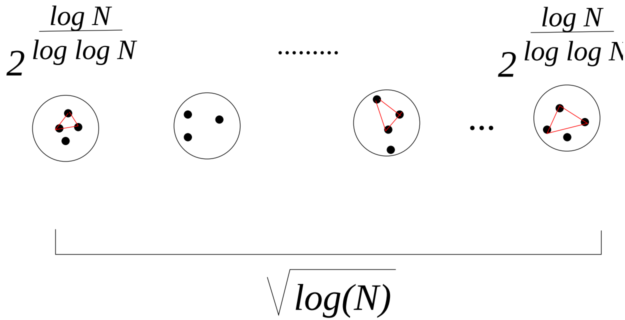

Apply induction in each small part.

Independent set of size $f(N) = 2^{(\log\log N)^2/\log\log\log N}$

$$f(N) = f\left(2^{\log N/\log\log N}\right)\sqrt{\log N}.$$

Hence

$$N^c \leq R_3^{semi}(4, n) \leq 2^{n^{o(1)}}.$$

Thank you!