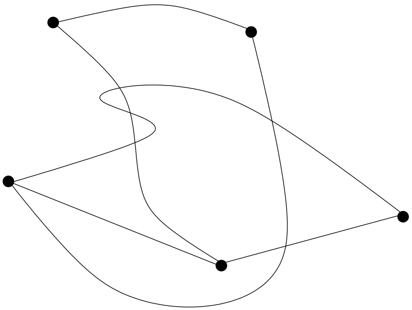
Extremal problems in topological graph theory

Andrew Suk

October 17, 2013

Definition

A *topological graph* is a graph drawn in the plane with vertices represented by points and edges represented by curves connecting the corresponding points. A topological graph is *simple* if every pair of its edges intersect at most once.



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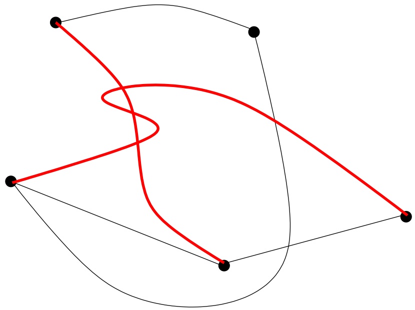
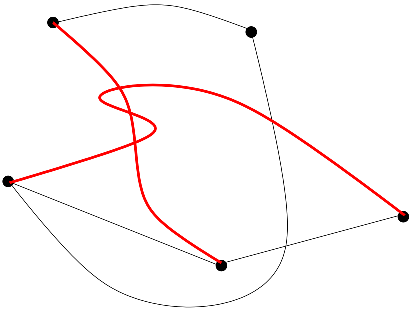


Crossing edges

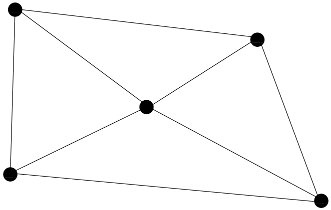
Two edges in a topological graph **cross** if they have a common interior point.



Application of Euler's Polyhedral formula:

Theorem

Every n-vertex topological graph with no crossing edges contains at most $3n - 6 = O(n)$ edges.

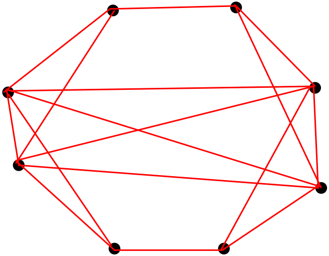


Relaxation of planarity.

Conjecture

Every n-vertex topological graph with no k pairwise crossing edges contains at most $O(n)$ edges.

All such graphs are called *k-quasi-planar*.



Conjecture

Every n-vertex k-quasi-planar graph has at most $O(n)$ edges.

Generated a lot of research, 1990's - present, different variations.

Conjecture has been proven for

1. $k = 3$ by Pach, Radoičić, Tóth 2003, Ackerman and Tardos 2007.
2. $k = 4$ by Ackerman 2008.

Open for $k \geq 5$.

Best known bound for $k \geq 5$

Theorem (Pach, Radoičić, Tóth 2003)

Every n-vertex k-quasi-planar graph has at most $n(\log n)^{4k-12}$ edges.

As an application of a separator Theorem by Matoušek 2013:

Theorem (Fox and Pach 2013)

Every n-vertex k-quasi-planar graph has at most $n(\log n)^{O(\log k)}$ edges.

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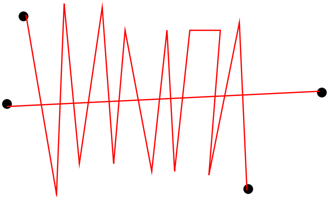
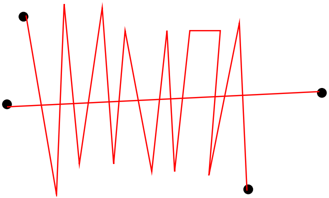


Two edges may cross n^n times.

Best known bound for $k \geq 5$

Theorem (Fox and Pach 2013)

Every n-vertex k-quasi-planar graph has at most $n(\log n)^{O(\log k)}$ *edges.*



Suk and Walczak 2013: We improve this bound in two special cases.

Two result from

A. Suk, B. Walczak, New bounds on the maximum number of edges in k-quasi-planar graphs, 21st International Symposium on Graph Drawing (GD '13). Bordeaux France, 2013.

Special Case 1

- G is an n-vertex k-quasi planar graph,
- **extra condition:** every pair of edges have at most t (say 1000) points in common.
- $|E(G)| \leq n(\log n)^{O(\log k)}$, Fox and Pach 2008

Theorem (Suk and Walczak 2013)

Every n-vertex k-quasi-planar graph with no two edges having more than t points in common, has at most $c_{k,t}2^{\alpha(n)}(n \log n)$ edges.

$\alpha(n)$ denotes the inverse Ackermann function (very slow).

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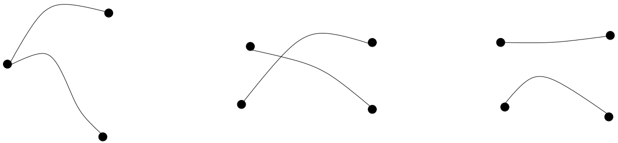
Main tool: A Theorem on Genearlized Davenport-Schinzel sequences.

Special Case 2

G is a **simple** k-quasi-planar graph:



1. $|E(G)| \leq n(\log n)^{O(k)}$, Pach, Shahrokhi, Szegedy 1996.
2. $|E(G)| \leq n(\log n)^{O(\log k)}$, Fox and Pach 2008.
3. $|E(G)| \leq c_k 2^{\alpha(n)} n(\log n)$, Fox, Pach, Suk 2012.

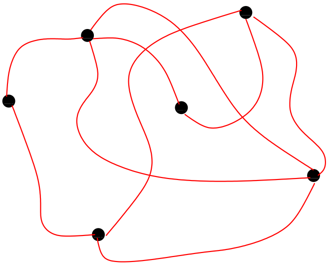
Special Case 2

$|E(G)| \leq c_k 2^{\alpha(n)} n(\log n)$, Fox, Pach, Suk 2012.
Using new/different methods:

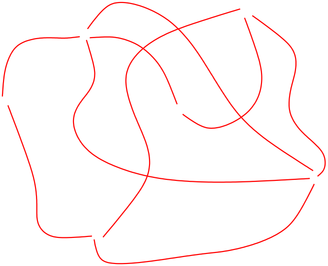
Theorem (Suk and Walczak 2013)

Every n-vertex **simple** *k-quasi-planar graph has at most $O(n \log n)$ edges.*

$G = (V, E)$ k-quasi-planar graph.



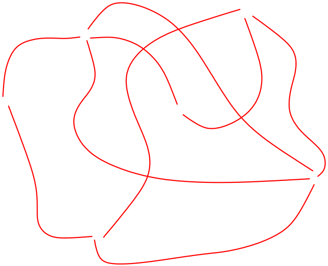
E is a family of $|E(G)|$ curves in the plane, no k pairwise intersecting.



Conjecture (Coloring conjecture)

Let F be a family of curves in the plane such that no k members pairwise intersect. Then $\chi(F) \leq c_k$.

Color the curves such that each color class consists of pairwise disjoint curves.



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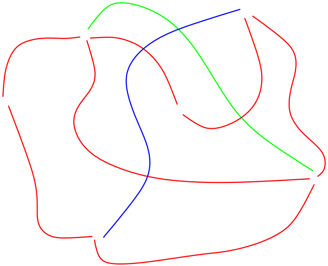

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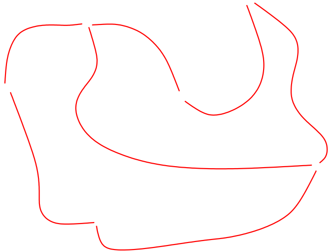

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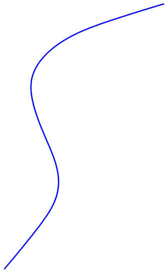

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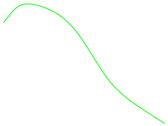

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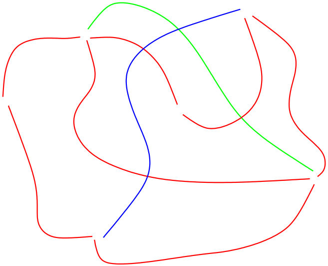
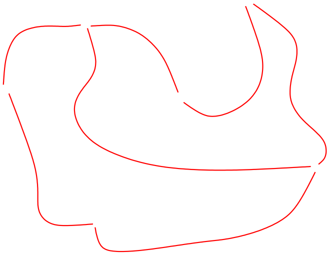


Conjecture (Coloring conjecture)

Let F be a family of curves in the plane such that no k members pairwise intersect. Then $\chi(F) \leq c_k$.

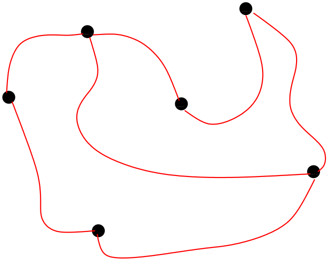
One of the color classes has at least $|E(G)|/c_k$ curves (edges).



Conjecture (Coloring conjecture)

Let F be a family of curves in the plane such that no k members pairwise intersect. Then $\chi(F) \leq c_k$.

$$\frac{|E(G)|}{c_k} \leq 3n - 6$$



Conjecture (Coloring conjecture, FALSE)

Let F be a family of curves in the plane such that no k members pairwise intersect. Then $\chi(F) \leq c_k$.

Conjecture is False!

Theorem (Pawlik, Kozik, Krawczyk, Lason, Micek, Trotter, Walczak, 2012)

For infinite values n, there exists a family F of n segments in the plane, no three members pairwise cross, and $\chi(F) > \Omega(\log \log n)$.

Conjecture (Coloring conjecture, FALSE)

Let F be a family of curves in the plane such that no k members pairwise intersect. Then $\chi(F) \leq c_k$.

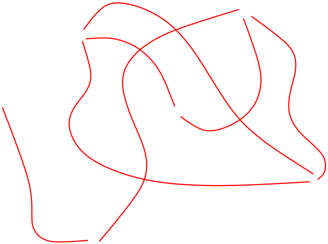
Conjecture true under extra conditions?

Theorem (Suk and Walczak, 2013)

Let F be a family of curves in the plane such that no k members pairwise intersect. Furthermore, suppose

1. *F is* **simple**,
2. *there is a curve β that intersects every member in F exactly once,*

then $\chi(F) \leq c_k$.



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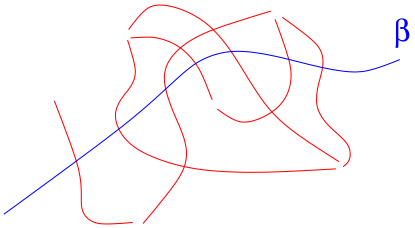

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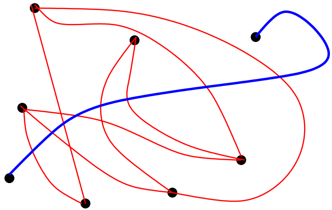
1. Coloring intersection graphs of arcwise connected sets in the plane, Lason, Micek, Pawlik and Walczak 2013.
2. Coloring intersection graphs of x-monotone curves in the plane, Suk 2012.
3. On bounding the chromatic number of L-graphs, McGuinness 1996.

Application of coloring result.

Corollary (Suk and Walczak, 2013)

For fixed $k > 1$, let G be a **simple** *n-vertex k-quasi planar graph. If G contains an edge that crosses every other edge, then $|E(G)| \leq O(n)$.*

G

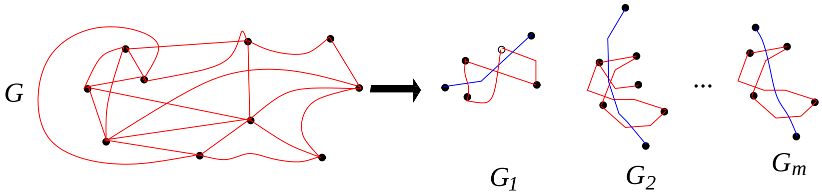


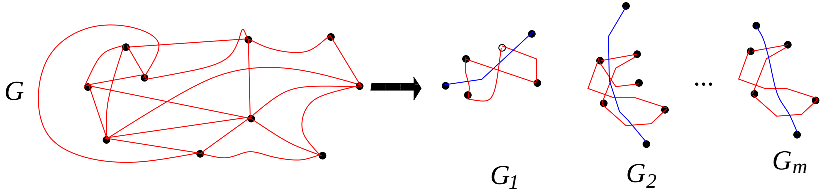
Lemma (Fox, Pach, Suk, 2012)

Let G be a simple topological graph on n vertices. Then there are subgraphs $G_1, G_2, \ldots, G_m \subset G$ such that

$$\frac{|E(G)|}{c \log n} \leq \sum_{i=1}^{m} |E(G_i)|,$$

every edge in G_i is disjoint to every edge in G_j. G_i has an edge that crosses every other edge in G_i.





Let $n_i = |V(G_i)|$.

- $|E(G_i)| \leq c_k n_i$, Suk and Walczak 2013.

$$\frac{|E(G)|}{c \log n} \leq \sum_{i=1}^{m} |E(G_i)| \leq \sum_{i=1}^{m} c_k n_i = c_k(n_1 + n_2 + \cdots + n_m) = c_k n.$$

□

Topological graph with no k pairwise crossing edges.

$|E(G)| \leq n(\log n)^{4k-12}$, Pach, Radoicic, Tóth.
$|E(G)| \leq n(\log n)^{O(\log k)}$ Fox, Pach.

Theorem (Suk and Walczak 2013)

Every n-vertex k-quasi-planar graph with no two edges having more than t points in common, has at most $c_{k,t} n(\log n)^{1+\epsilon}$ edges.

Theorem (Suk and Walczak 2013)

Every n-vertex **simple** *k-quasi-planar graph has at most $O(n \log n)$ edges.*

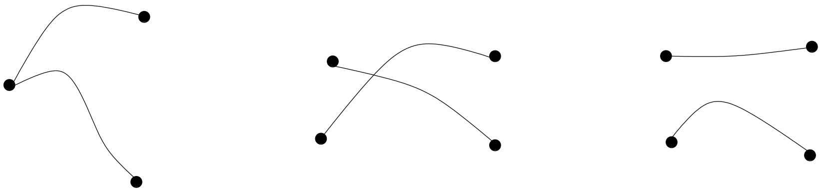
Goal: $|E(G)| \leq O(n)$.

Topological graphs with no k-pairwise **disjoint** edges?

Dual problem

Topological graphs with no *k*-pairwise **disjoint** edges?

We will only consider *simple* topological graphs (see why later).



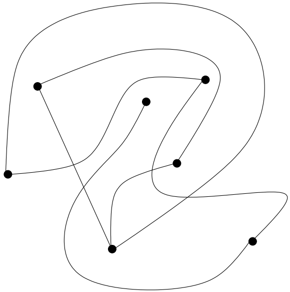
Conjecture (Conway)

Every n-vertex simple topological graph with no two disjoint edges, has at most n edges.

Theorem (Lovász, Pach, Szegedy, 1997)

Every n-vertex simple topological graph with no two disjoint edges, has at most $2n$ edges.

Best known $1.43n$ by Fulek and Pach, 2010.



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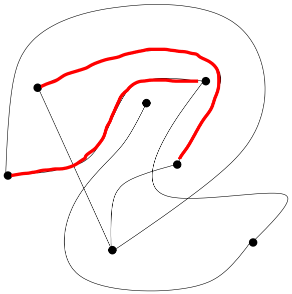


Generalization.

Theorem (Pach and Tóth, 2005)

Every n-vertex simple topological graph with no k pairwise disjoint edges, has at most $C_k n \log^{5k-10} n$ *edges.*

Conjecture to be at most $O(n)$ (for fixed k). By solving for k in $C_k n \log^{5k-10} n = \binom{n}{2}$.

Corollary (Pach and Tóth, 2005)

Every complete n-vertex simple topological graph has at least $\Omega(\log n / \log\log n)$ *pairwise disjoint edges.*

Conjecture (Pach and Tóth)

There exists a constant δ, such that every complete n-vertex simple topological graph has at least $\Omega(n^{\delta})$ pairwise disjoint edges.

History

Pairwise disjoint edges in complete n-vertex simple topological graphs:

1. $\Omega(\log^{1/6} n)$, Pach, Solymosi, Tóth, 2001.
2. $\Omega(\log n / \log\log n)$, Pach and Tóth, 2005.
3. $\Omega(\log^{1+\epsilon} n)$, Fox and Sudakov, 2008.

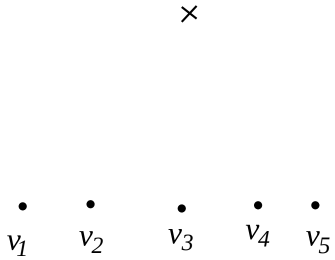
Note $\epsilon \approx 1/50$. All results are slightly stronger statements.

Pach and Tóth conjecture: **True**.

Theorem (Suk, 2012)

Every complete n-vertex simple topological graph has at least $\Omega(n^{1/3})$ pairwise disjoint edges.

Clearly the simple condition is required.



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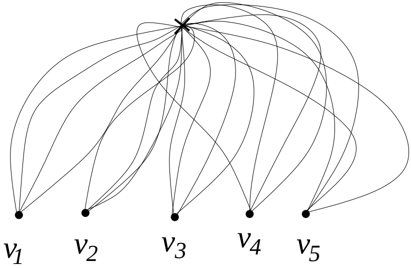

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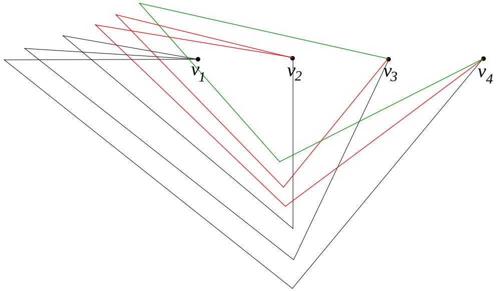


Every pair of edges cross **once** or **twice** (no more or less).

Theorem (Suk, 2012)

Every complete n-vertex simple topological graph has at least $\Omega(n^{1/3})$ pairwise disjoint edges.

Let $\mathcal{F}$ be a set system with ground set X.

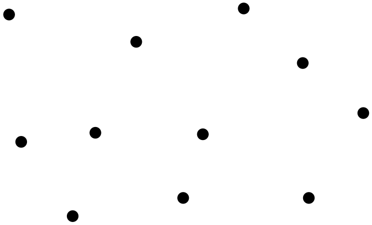
Definition (Dual shatter function)

The dual shatter function $\pi^*_{\mathcal{F}}(m)$, is defined to be the maximum number of equivalence classes on X, defined by an m-element subfamily of $\mathcal{F}$.

For m sets $S_1, S_2, ..., S_m$, $x \sim y$ if BOTH x, y are in exactly the same sets among $S_1, ..., S_m$ (i.e. no set S_i contains x and not y or vise versa).

I.e. $\pi^*_{\mathcal{F}}(m)$ is the number of nonempty cells in the Venn diagram of m sets of $\mathcal{F}$.

Example: X is a set of n points in the plane, $\mathcal{F}$ is the set of all halfplanes.



$\pi^*_{\mathcal{F}}(m) = O(m^2)$.

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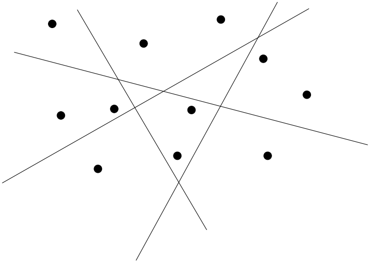

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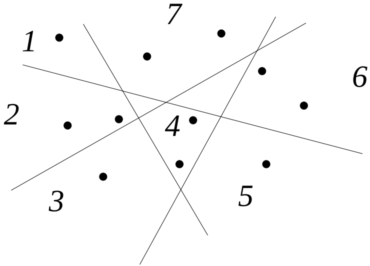

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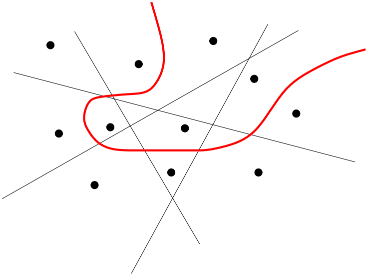

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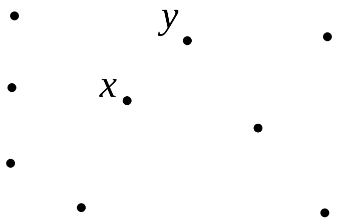
definition

A set $S \in \mathcal{F}$ *stabs* the pair (of vertices) x, y if $|S \cap \{x, y\}| = 1$.



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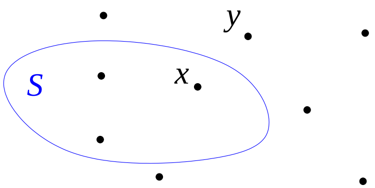

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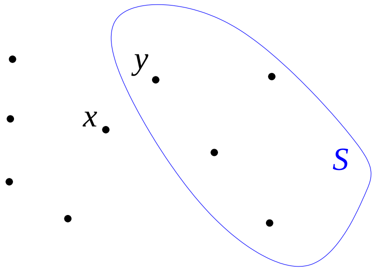

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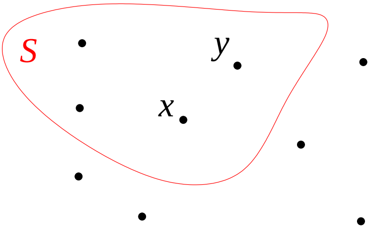
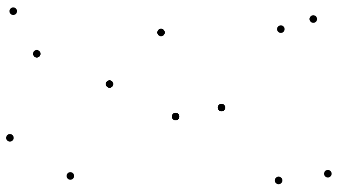


Main tool

Theorem (Matching theorem, Chazelle and Welzl, 1989)

Let $\mathcal{F}$ be a set system on an n element point set X (n is even), such that $\pi^*_{\mathcal{F}}(m) \leq O(m^d)$. Then there exists a perfect matching M on X such that each set in $\mathcal{F}$ stabs at most $O(n^{1-1/d})$ members in M.



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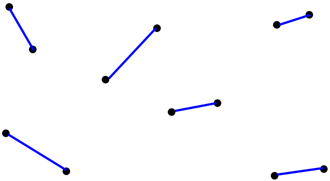

$$M = \{(x_1, y_1), (x_2, y_2), \ldots, (x_{n/2}, y_{n/2})\}.$$

Main tool

Theorem (Chazelle and Welzl, 1989)

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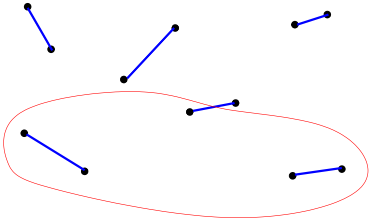

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Main tool

Theorem (Matching Lemma, Chazelle and Welzl 1989)

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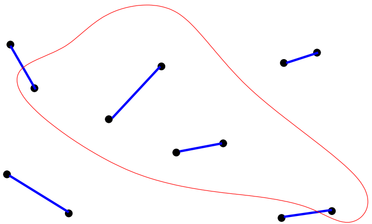

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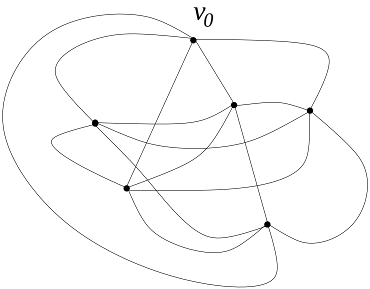
Sketch of proof

Theorem (Suk, 2012)

Every complete n-vertex simple topological graph has at least $\Omega(n^{1/3})$ pairwise disjoint edges.

K_{n+1}



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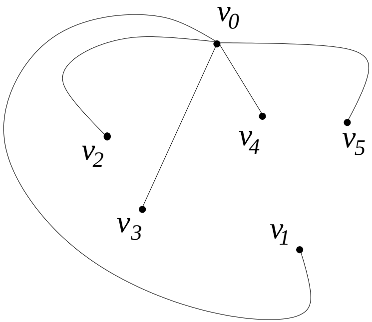
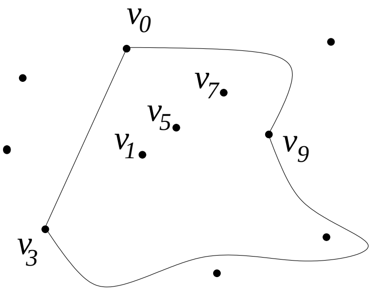


Define $\mathcal{F}_1 = \bigcup_{1 \leq i < j \leq n} \{S_{i,j}\}$, where $S_{i,j}$ is the set of vertices inside triangle v_0, v_i, v_j.



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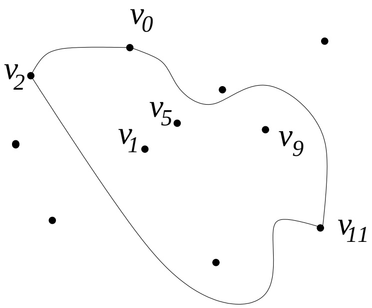

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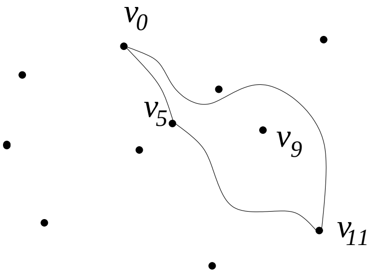


$S_{3,9} = \{v_1, v_5, v_7\}$, $S_{2,11} = \{v_1, v_5, v_9\}$, $S_{5,11} = \{v_9\}$.

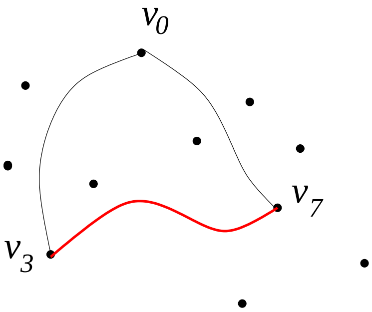
$\mathcal{F}_1$ is not "complicated".

Lemma

$\pi^*_{\mathcal{F}_1}(m) \leq O(m^2)$.

Proof: Basically m "triangles" divides the plane into at most $O(m^2)$ regions. Proof is by induction on m.

Define set system $\mathcal{F}_2 = \bigcup_{1 \leq i < j \leq n} \{S'_{i,j}\}$, where $v_k \in S'_{i,j}$ if topological edges $v_0 v_k$ and $v_i v_j$ cross.



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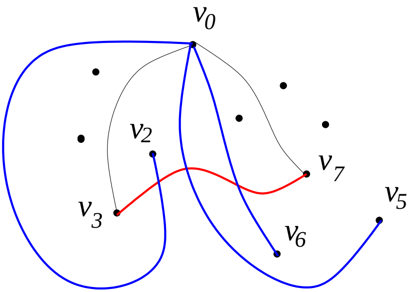

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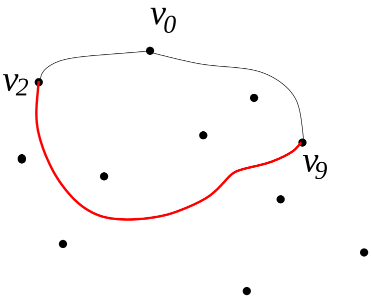
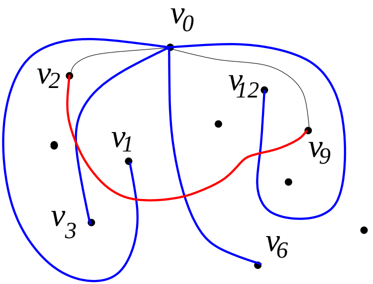


$S'_{3,7} = \{v_2, v_6, v_5\}$, $S'_{2,9} =?$.

Define set system $\mathcal{F}_2 = \bigcup_{1 \leq i < j \leq n} \{S'_{i,j}\}$, where $v_k \in S'_{i,j}$ if topological edges $v_0 v_k$ and $v_i v_j$ cross.



$S'_{3,7} = \{v_2, v_6, v_5\}, S'_{2,9} = \{v_1, v_3, v_6, v_{12}\}$.

Again, $\mathcal{F}_2$ is not "complicated". Set $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. One can show

Lemma

$\pi^*_{\mathcal{F}}(m) = O(m^3)$.

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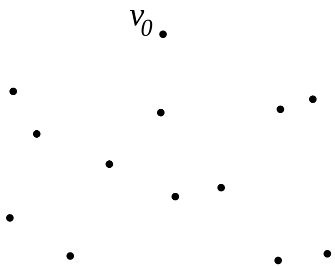

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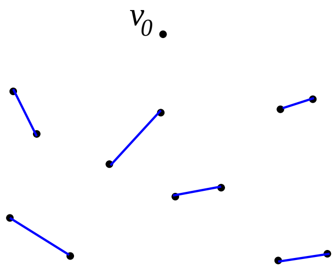

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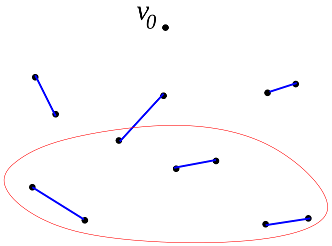
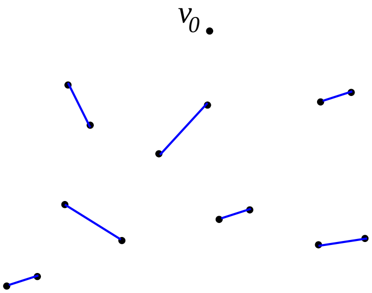


Auxiliary graph G, where $V(G) = M$ and $v_iv_j \to v_kv_l$ if $S_{i,j}$ or $S'_{i,j}$ stabs v_kv_l.



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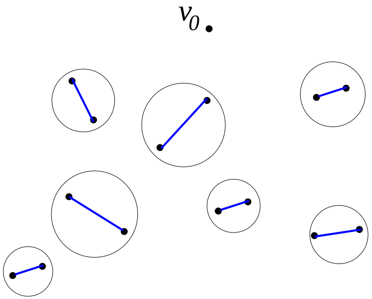

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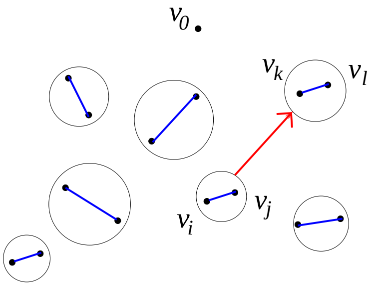

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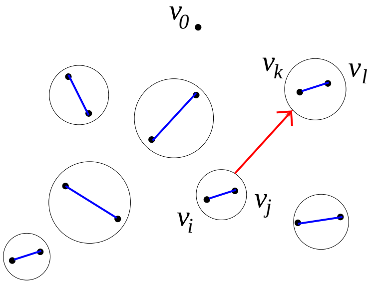
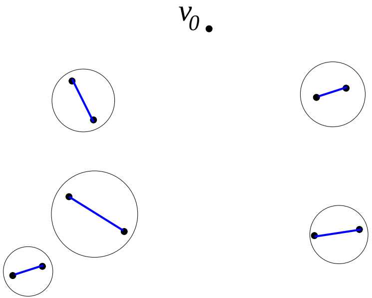


$S_{i,j}$ and $S'_{i,j}$ stabs (in total) at most $O(n^{2/3})$ members in $M = V(G)$. $|E(G)| \leq O(n^{5/3})$.

$|E(G)| \leq O(n^{5/3})$, by Turán, G contains an independent set of size $\Omega(n^{1/3})$.



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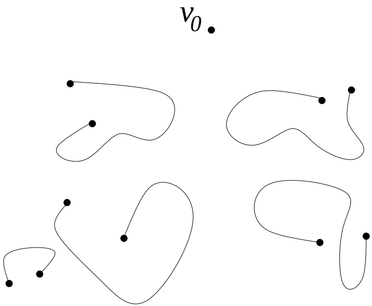

Claim!

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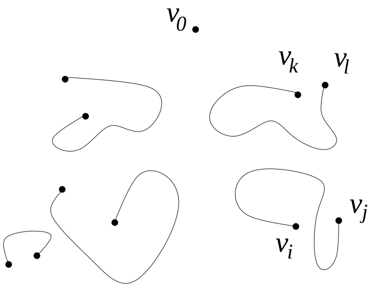
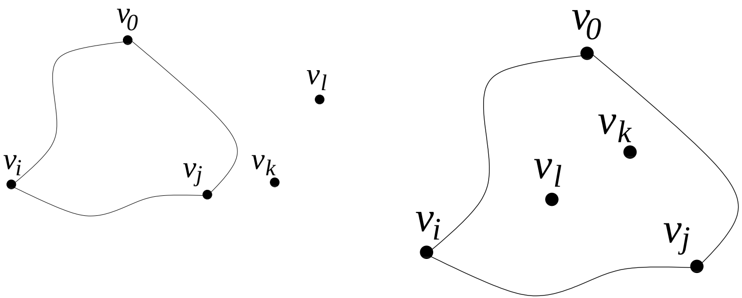
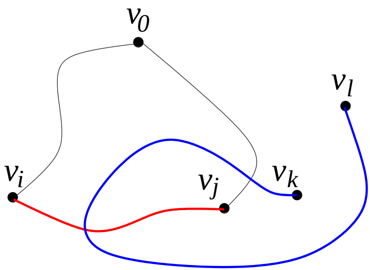


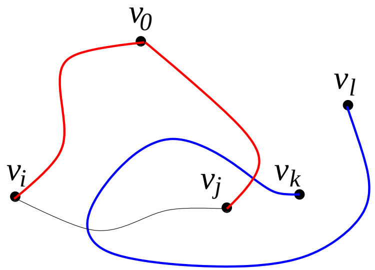
Claim!

Since $S_{i,j}$ does NOT stab $v_k v_l$

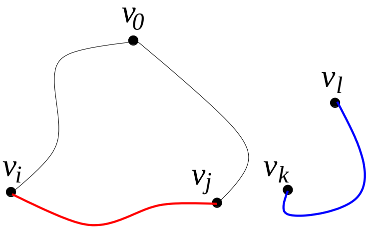




Assume edges cross.

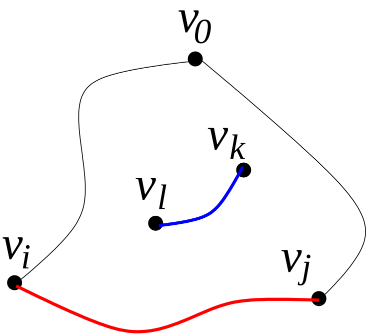


$S'_{k,l}$ stabs $v_i v_j$, which is a contradiction.

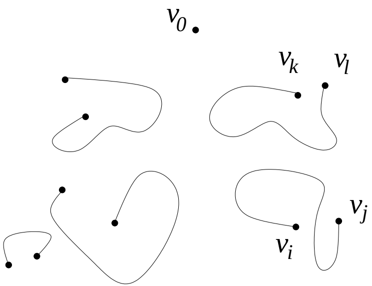


Two edges must be disjoint.

Same argument shows



$\Omega(n^{1/3})$ pairwise disjoint edges in K_{n+1}.



Open Problems.

1. Best known upper bound construction: $O(n)$ pairwise disjoint edges.
2. Find $\Omega(n^{\delta})$ pairwise disjoint edges in dense simple topological graphs.

Conjecture (Coloring conjecture, FALSE)

Let F be a family of curves in the plane such that no k members pairwise intersect. Then $\chi(F) \leq c_k$.

False.

Conjecture

Let F be a family of curves in the plane such that no k members pairwise intersect. Then F contains $\frac{|F|}{c_k}$ pairwise disjoint members.

Open!

Problem: Let F be a family of lines in $\mathbb{R}^3$, such that no 3 members pairwise intersect. Then is $\chi(F) \leq c$?

Thank you!