Density-type theorems for semi-algebraic hypergraphs

Jacob Fox MIT, Janos Pach EPFL, Andrew Suk UIC

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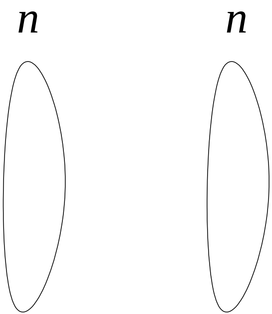
An old Ramsey-type result, Kövári, Sós, and Turán and Erdős

Bipartite graph G, edge set E.



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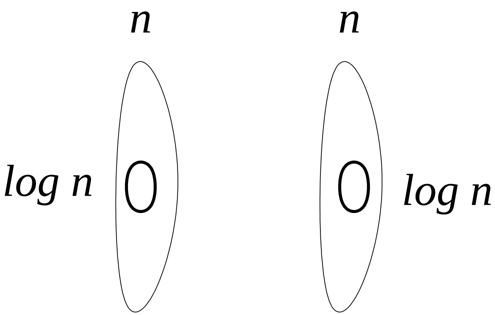

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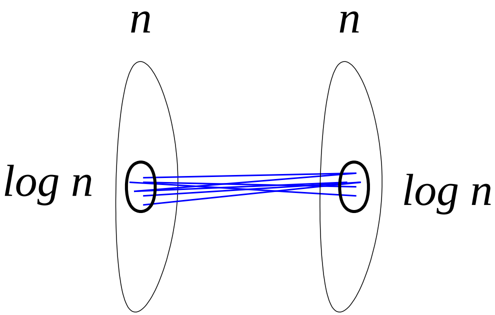
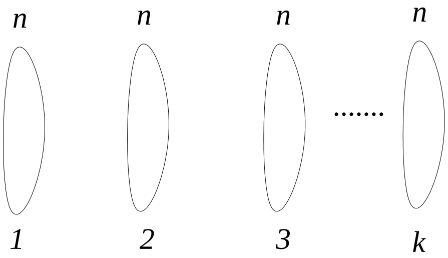


Generalization, Erdős

k-partite k-uniform hypergraph H, edge set E.



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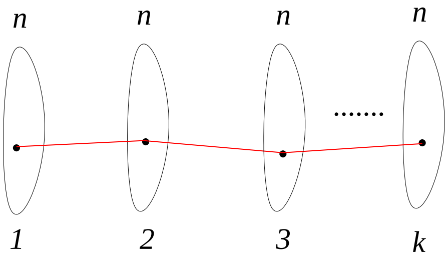

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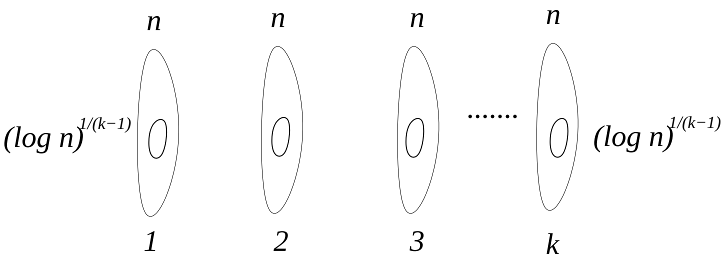

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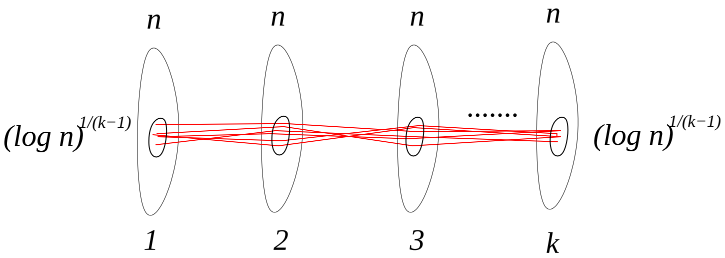

These results are tight.

Generalization, Erdős

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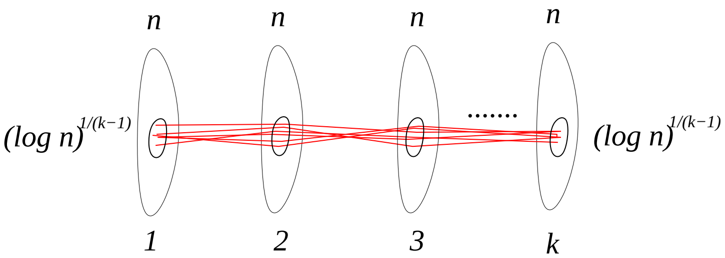
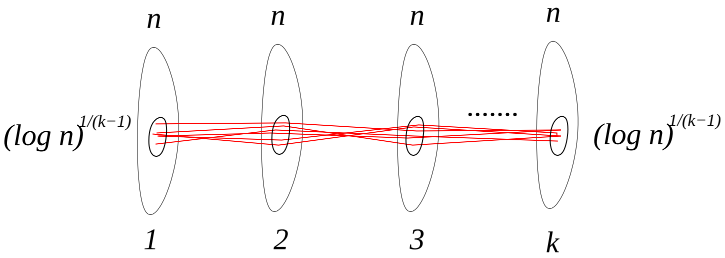


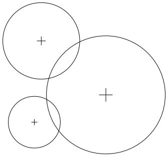
In this talk: We can do much better if H is a semi-algebraic k-uniform hypergraph.

k-partite k-uniform hypergraph H, edge set E.



Semi-algebraic hypergraphs: $V = \{$simple geometric objects in $\mathbb{R}^d\}$, $E = \{$simple relation on k tuples of $V\}$.

$V = \{A_1, ..., A_n\}$, n disks in the plane. $E = \{$pairs of disks that intersect$\}$.



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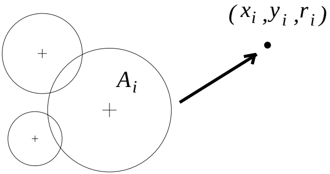

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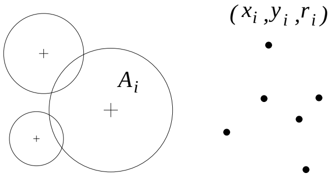

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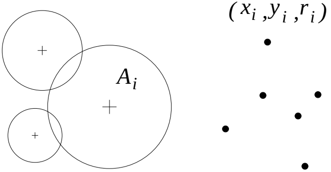

$A_i \rightarrow p_i = (x_i, y_i, r_i)$, $A_j \rightarrow p_j = (x_j, y_j, r_j)$. A_i and A_j cross if and only if

$$-x_i^2 + 2x_ix_j - x_j^2 - y_i^2 + 2y_iy_j - y_j^2 + r_i^2 + 2r_ir_j + r_j^2 \geq 0.$$

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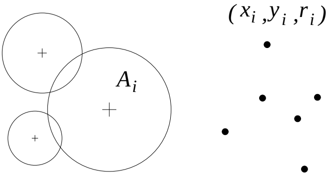


Graph $G = (V, E)$, $V = n$ points in $\mathbb{R}^3$

E defined by the polynomial

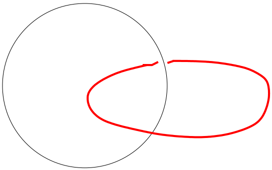
$$f(z_1, ..., z_6) = -z_1^2 + 2z_1z_4 - z_4^2 - z_2^2 + 2z_2z_5 - z_5^2 + z_3^2 + 2z_3z_6 + z_6^2.$$

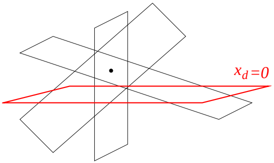
$$(p_i, p_j) \in E \Leftrightarrow f(p_i, p_j) \geq 0.$$

Examples

1. $V = \{n \text{ circles in } \mathbb{R}^3\}$
   $E = \{\text{pairs that are linked}\}$.

2. $V = \{n \text{ hyperplanes in } \mathbb{R}^d \text{ in general position}\}$,
   $E = \{d\text{-tuples whose intersection point is above the hyperplane } x_d = 0\}$.





More examples of semi-algebraic hypergraphs

Examples

1. $V = \{n \text{ circles in } \mathbb{R}^3\} \rightarrow$ n points in higher dimensions.
   $E = \{\text{pairs that are linked}\} \rightarrow$ polynomials $f_1, ..., f_t$.
2. $V = \{n \text{ hyperplanes in } \mathbb{R}^d \text{ in general position}\} \rightarrow$ n points in higher dimensions,
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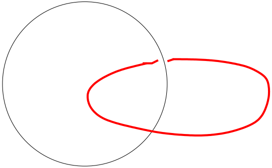
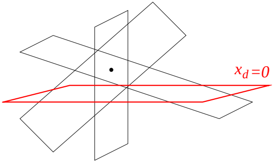




We say that $H = (V, E)$ is a **semi-algebraic k-uniform hypergraph in d-space** if

$V = \{n \text{ points in } \mathbb{R}^d\}$

E defined by polynomials $f_1, \ldots, f_t$ and a Boolean formula Φ such that

$$(p_{i_1}, \ldots, p_{i_k}) \in E$$

$$\Leftrightarrow \Phi(f_1(p_{i_1}, \ldots, p_{i_k}) \geq 0, \ldots, f_t(p_{i_1}, \ldots, p_{i_k}) \geq 0) = \text{yes}$$

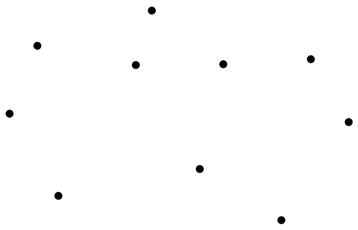
Example

3-uniform hypergraph $H = (V, E)$, $V = \{p_1, ..., p_n\}$ points in $\mathbb{R}^d$.

Relation $E \subset \binom{V}{3}$ depends on f and Φ

$$\phi(f(x_1, x_2, x_3) \geq 0) = \{\text{yes,no}\}$$

$(p_1, p_2, p_3) \in E$ depends on $f(p_1, p_2, p_3) \rightarrow \{+, -, 0\}$.



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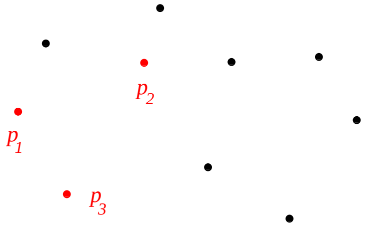

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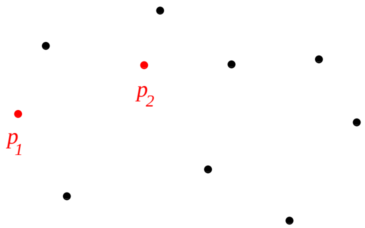
Zero set $f(p_1, p_2, x_3) = 0$, surface in $\mathbb{R}^d$.



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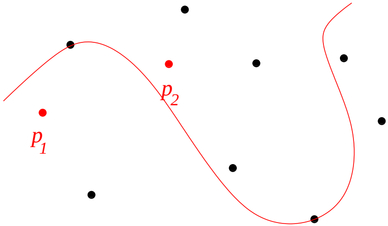

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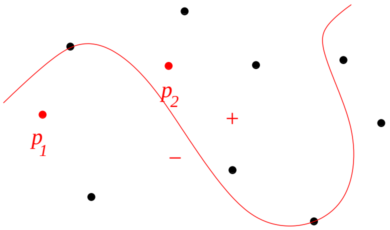


E has complexity (t, D), degree of $f(p_1, p_2, x_3) \leq D$.

Complexity of relation E

$x_i \in \mathbb{R}^d$

E has complexity (t, D)

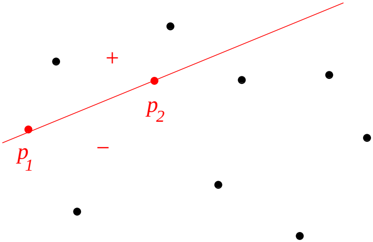
1. described by polynomials $f_1, ..., f_t$,
2. and the degree of ALL kt d-variate polynomials $f_i(x_1, ..., x_{k-1}, x_k), f_i(x_1, ..., x_{k-2}, x_{k-1}, x_k), ..., f_i(x_1, x_2, ..., x_k)$, for $i = 1...t$, is at most D.

Note. f_i has degree at most Dk.

Motivation: Orientations and order-types

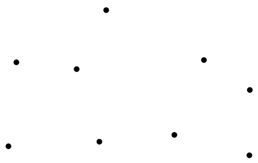
Our motivation, E is related to order-types and orientations.

$f(p_1, p_2, x_3)$ is linear.



E has complexity $(t, D) = (t, 1)$.

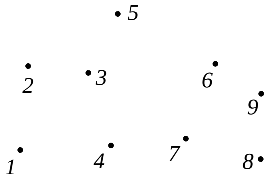
Example



$V = \{n \text{ points in the plane}\}$,
$E = \{\text{triples having a clockwise orientation}\}$.

$H = (V, E)$ semi-algebraic 3-uniform hypergraph in the plane $(d = 2)$

Example



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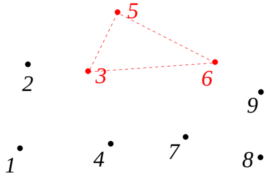

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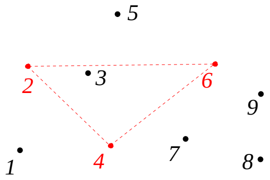

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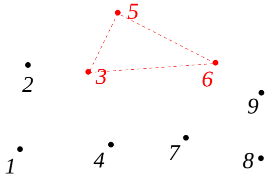
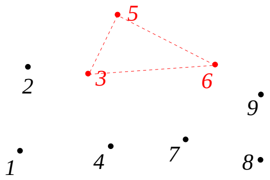


$E = \{\text{triples having a clockwise orientation}\}.$

$$det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} > 0.$$

Example



Complexity of E is (t, D), where $t = 1, D = 1$

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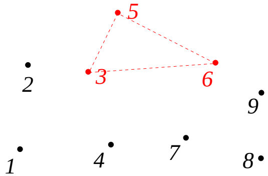

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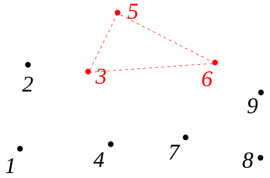

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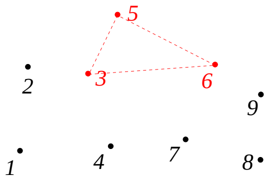

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Example in higher dimensions

$E = (d+1)$-tuples with a positive orientation, complexity $(t, D) = (1, 1)$.

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ | & | & \cdots & | \\ p_{i_1} & p_{i_2} & \vdots & p_{i_{d+1}} \\ | & | & \cdots & | \end{pmatrix} > 0.$$

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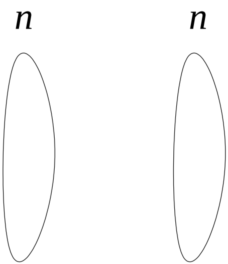
$$\det \begin{pmatrix} 1 & \color{red}{1} & \cdots & \color{red}{1} \\ | & \color{red}{|} & \cdots & \color{red}{|} \\ p_{i_1} & \color{red}{p_{i_2}} & \color{red}{\vdots} & \color{red}{p_{i_{d+1}}} \\ | & \color{red}{|} & \cdots & \color{red}{|} \end{pmatrix} > 0.$$

Previous results

Theorem (Alon, Pach, Pinchasi, Radoicic, Sharir 2005)

Let $H = (V_1, V_2, E)$ be a bipartite semi-algebraic graph ($k = 2$) in d-space, where $|V_1| = |V_2| = n$ and E has complexity (t, D). Then there are subsets $V_1', V_2' \subset V$ such that $|V_i'| \geq \epsilon n$ and either $(V_1', V_2') \subset E$ or $(V_1', V_2') \subset \overline{E}$, and $\epsilon = \epsilon(d, t, D)$.



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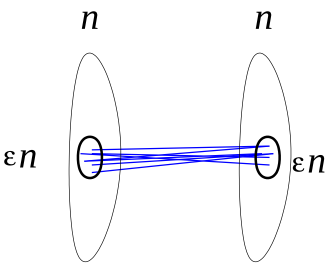

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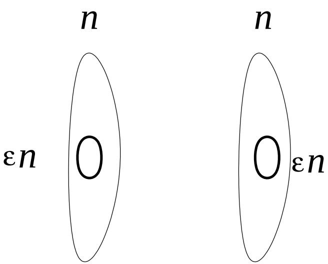

Stronger density theorem

Including an argument of Komlos:

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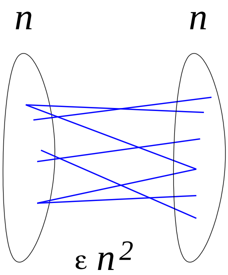
Let $H = (V_1, V_2, E)$ be a bipartite semi-algebraic graph ($k = 2$) in d-space, where $|V_1| = |V_2| = n$ and E has complexity (t, D). If $|E| \geq \epsilon n^2$, then there are subsets $V_1', V_2' \subset V$ such that $|V_i'| \geq \epsilon^C n$ where $C = C(d, t, D)$, and $(V_1', V_2') \subset E$.



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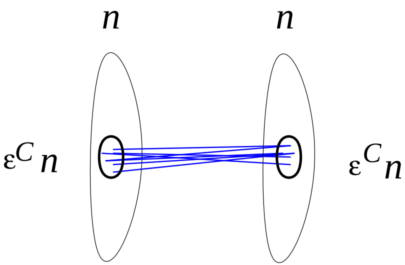
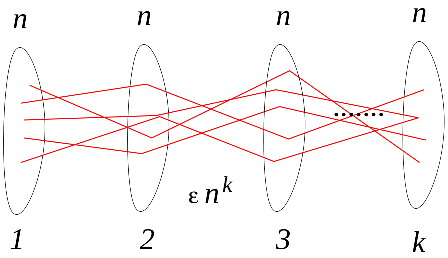


Theorem (Fox, Gromov, Lafforgue, Naor, Pach 2012, Bukh and Hubard 2012)

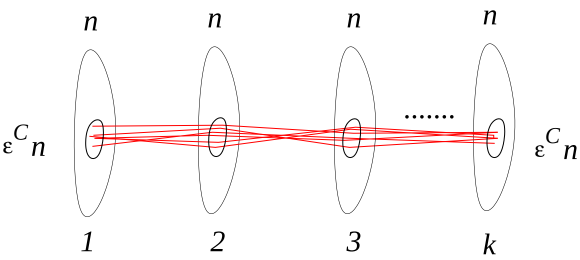
Let $H = (V_1, ..., V_k, E)$ be a k-partite semi-algebraic k-uniform hypergraph in d-space, where $|V_1| = \cdots = |V_k| = n$ and E has complexity (t, D). If $|E| \geq \epsilon n^k$, then there are subsets $V_1', ..., V_k' \subset V$ such that $|V_i'| \geq \epsilon^C n$ where $C = C(k, d, t, D)$, and $(V_1', ..., V_k') \subset E$.



Generalization

Theorem (Fox, Gromov, Lafforgue, Naor, Pach 2012, Bukh and Hubard 2012)

Let $H = (V_1, ..., V_k, E)$ be a k-partite semi-algebraic k-uniform hypergraph in d-space, where $|V_1| = \cdots = |V_k| = n$ and E has complexity (t, D). If $|E| \geq \epsilon n^k$, then there are subsets $V_1', ..., V_k' \subset V$ such that $|V_i'| \geq \epsilon^C n$ where $C = C(k, d, t, D)$, and $(V_1', ..., V_k') \subset E$.

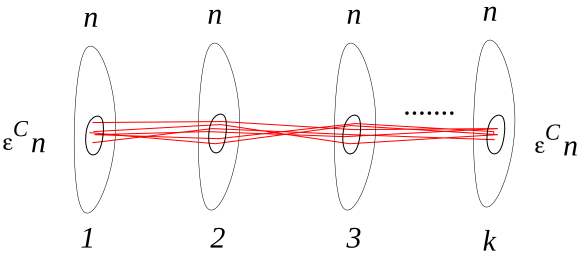


$C(k, d, t, D)$: Dependency on uniformity k is very bad.

Fox, Gromov, Lafforgue, Naor, Pach: $C(k, d, t, D) \sim \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2^d}}}}}}_{k}$

(tower-type)

Bukh-Hubard: $C(k, d, t, D) \sim 2^{2^{k+d}}$, double exponential in k.



Bukh-Hubard: Set sizes decay triple exponentially in k

New results

For simplicity, complexity (t, D) is fixed.

Theorem (Fox, Pach, Suk, 2013+)

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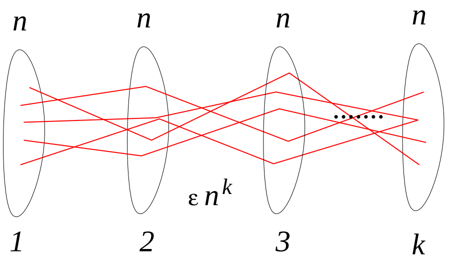

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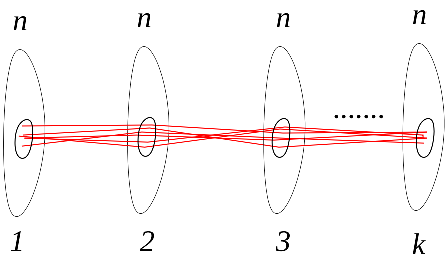

New results, $D = 1$

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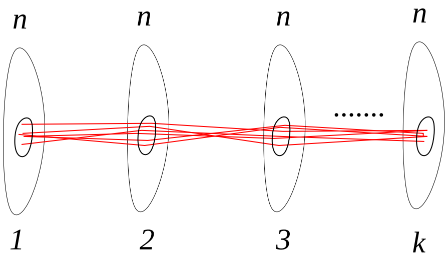


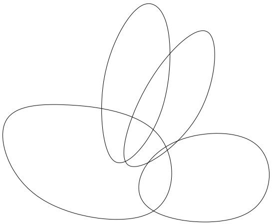
Applications, Tverberg-type result

Theorem (Pach, 1998)

Let $P_1, P_2, ..., P_{d+1} \subset \mathbb{R}^d$ be disjoint n-element point sets with $P_1 \cup \cdots \cup P_{d+1}$ in general position. Then there is a point $q \in \mathbb{R}^d$ and subsets $P_1' \subset P_1, ..., P_{d+1}' \subset P_{d+1}$, with

$$|P_i'| \geq 2^{-2^{2^{O(d)}}} n,$$

such that all closed rainbow simplices contains q.



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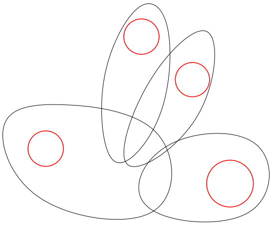

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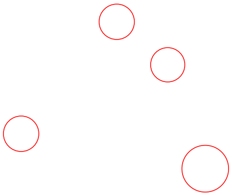

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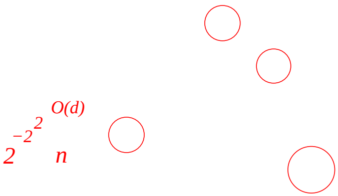

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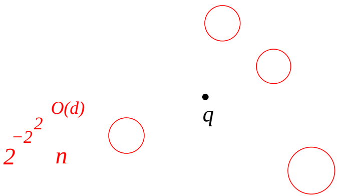

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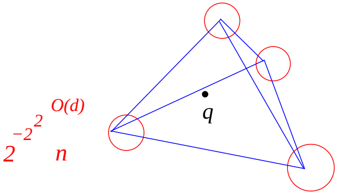

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Theorem (Bárány and Valtr, 1998)

Let $P_1, \ldots, P_k$ be n-element point sets in $\mathbb{R}^d$ such that $P_1 \cup \cdots \cup P_k$ is in general position. Then there are subsets $P'_1 \subset P_1, \ldots, P'_k \subset P_k$ such that the k-tuple $(P'_1, \ldots, P'_k)$ has same-type transversals and

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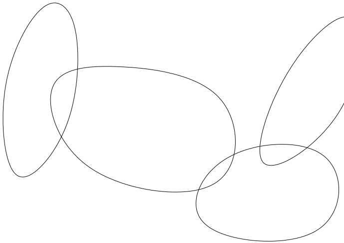
for all i.



Applications, Same-type Lemma

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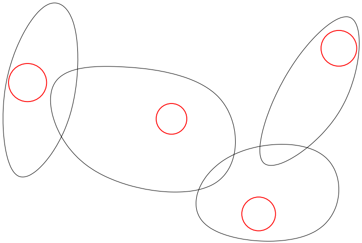

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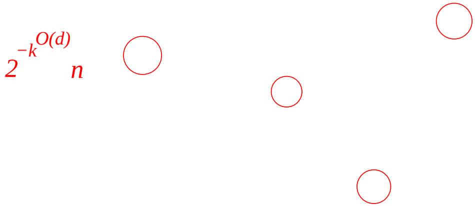

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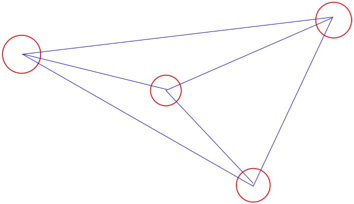

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Sketch proof of Same-type lemma

Theorem (Fox, Pach, Suk, 2013+)

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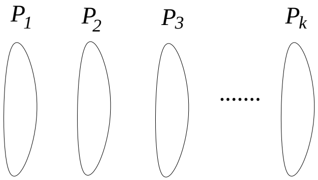

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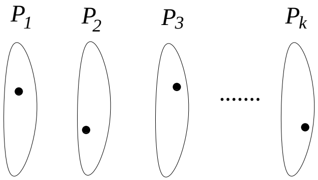
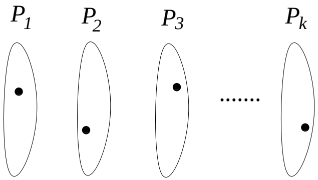
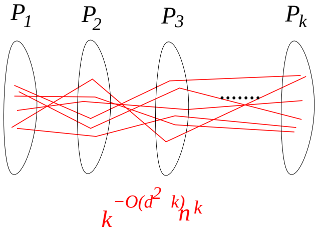




Goodman-Pollack: Number of different order-types of k-element point sets in d dimensions is at most $k^{O(d^2 k)}$.

There exists an order type π, such that at least $k^{-O(d^2 k)} n^k$ (rainbow) k-tuples have order type π.



k-partite k-uniform semi-algebraic hypergraph $H = (P_1, ..., P_k, E)$ in d-space

$E = \{k-\text{tuples with order type } \pi\}$. $|E| \geq k^{-O(d^2 k)} n^k$.

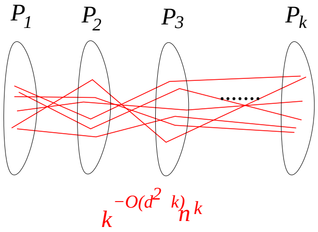
Complexity of E?

To check if $(p_1, ..., p_k)$ has order π, just check the orientation of each $(d+1)$-tuple. For each $p_{i_1}, ..., p_{i_{d+1}}$

$$\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ | & | & \cdots & | \\ p_{i_1} & p_{i_2} & \vdots & p_{i_{d+1}} \\ | & | & \cdots & | \end{pmatrix} \to \{+, -\}.$$

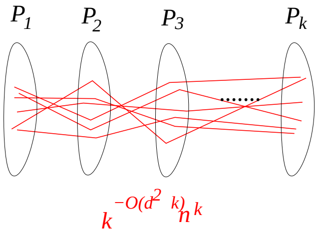
Hence we need to check $t = \binom{k}{d+1}$ polynomial inequalities.

Complexity of E is $(t, D) = (\binom{k}{d+1}, 1)$.



k-partite k-uniform semi-algebraic hypergraph $H = (P_1, ..., P_k, E)$ in d-space.

$\epsilon = k^{-O(d^2 k)}$, $t = \binom{k}{d+1}, D = 1$



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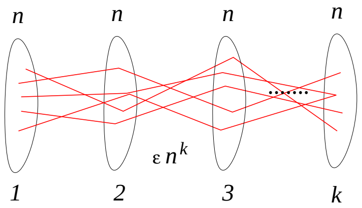
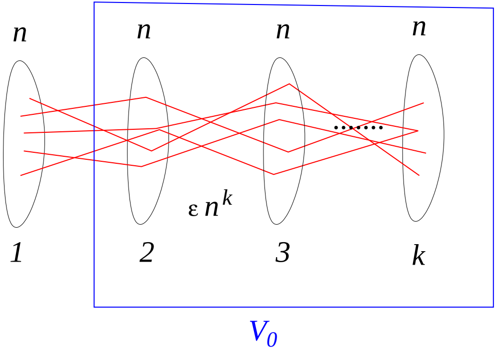
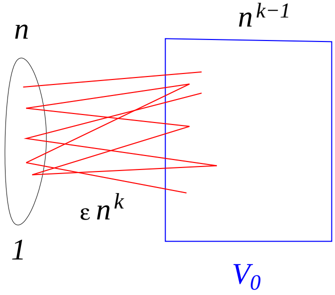
Proof. Induction on k. E depends on $f(\mathbb{R}^d, \mathbb{R}^d, ..., \mathbb{R}^d) \geq 0$.



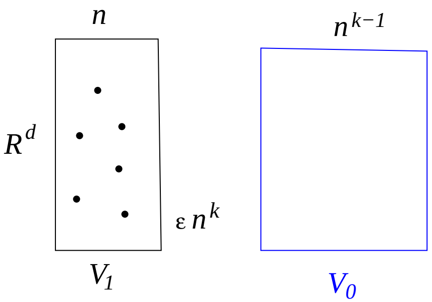
$V_0 = V_2 \times V_3 \times \cdots \times V_k \subset \mathbb{R}^{(k-1)d}$. $|V_0| = n^{k-1}$.



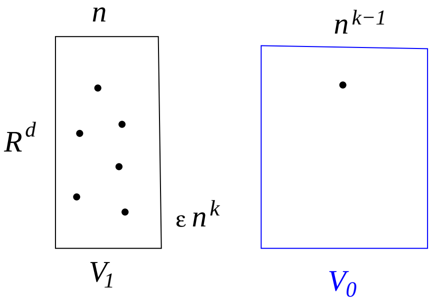
Bipartite graph $G = (V_1, V_0, E)$, E depends on $f(\mathbb{R}^d, \mathbb{R}^{(d-1)k}) \geq 0$.



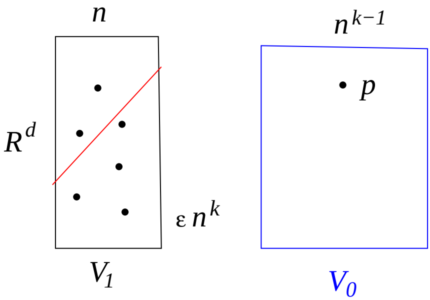
$V_1 \subset \mathbb{R}^d$.



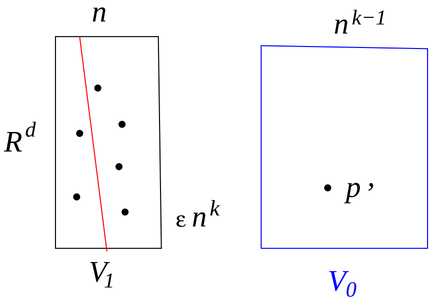
For each $p \in V_0 \subset \mathbb{R}^{(d-1)k}$



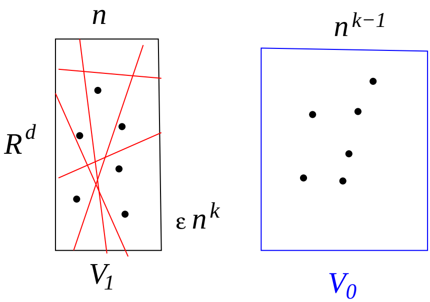
For each $p \in V_0 \subset \mathbb{R}^{(d-1)k}$, hyperplane $f(\mathbb{R}^d, p) = 0$ in $\mathbb{R}^d$.



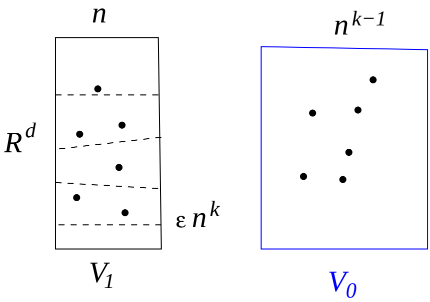
$p' \in V_0 \subset \mathbb{R}^{(d-1)k}$, hyperplane $f(\mathbb{R}^d, p') = 0$ in $\mathbb{R}^d$.



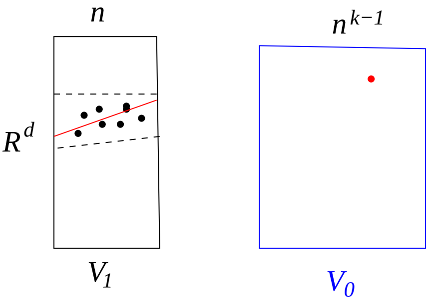
$H = \{n^{k-1} \text{ hyperplanes in } \mathbb{R}^d\}$.



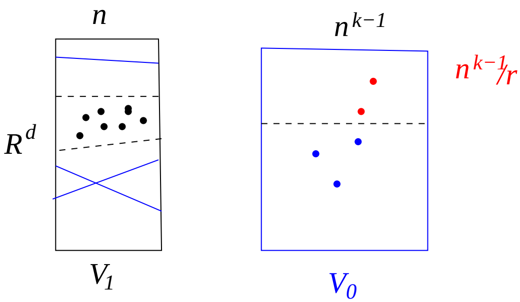
Cutting Lemma (Chazelle, Friedman 1990). For $r > 0$, Subdivide $\mathbb{R}^d$ into at most $2^{10d \log d} r^d$ simplices, such that at most n^{k-1}/r hyperplanes from H crosses each cell.



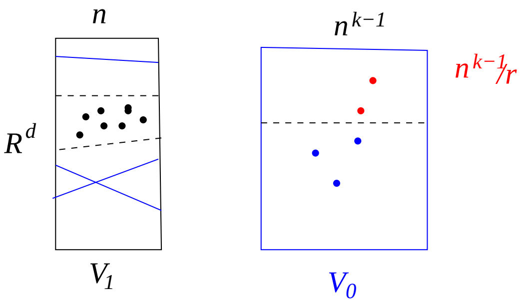
At most $\frac{n^{k-1}}{r}$ hyperplanes crosses Δ. $f(\mathbb{R}^d, p) = 0$



Most hyperplanes $f(\mathbb{R}^d, p) = 0$ do not cross Δ.



If hyperplane $f(\mathbb{R}^d, p) = 0$ does not cross Δ, then sign pattern does not change.



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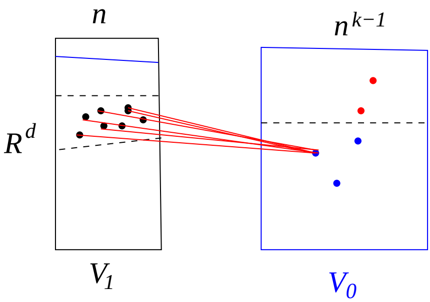
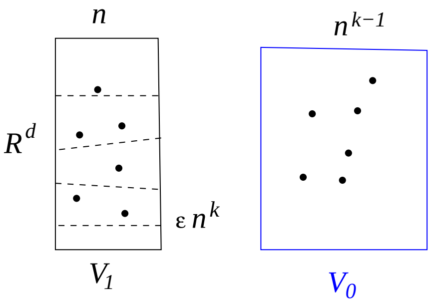
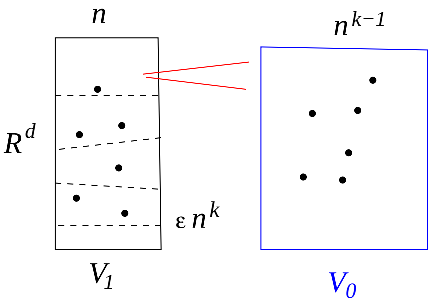


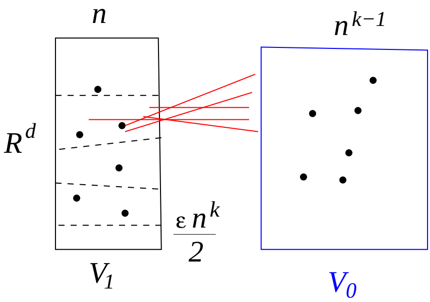
Divided $\mathbb{R}^d$ into $2^{10d \log d} r^d$ cells, on average a cell has $\frac{n}{2^{10d \log d} r^d}$ points.



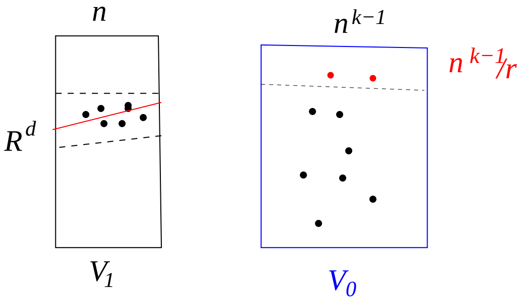
If a cell has fewer than $\frac{n}{2^{10d \log d} r^d}(\epsilon/2)$ points, DELETE all edges emanating out of it. Still have a $(\epsilon/2)n^k$ edges



Cells with edges has at least $\frac{n}{2^{10d \log d} r^d}(\epsilon/2)$ points. Still have a $(\epsilon/2)n^k$ edges



Each "big" cell gives rise to a certain number of vertices in V_0 that is adjacent to all points in it.



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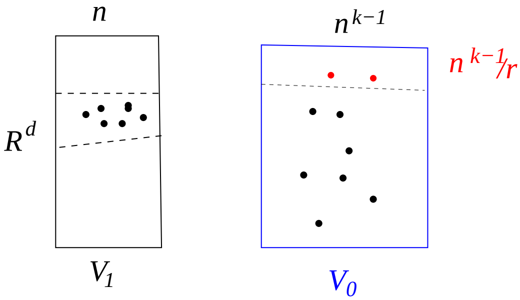

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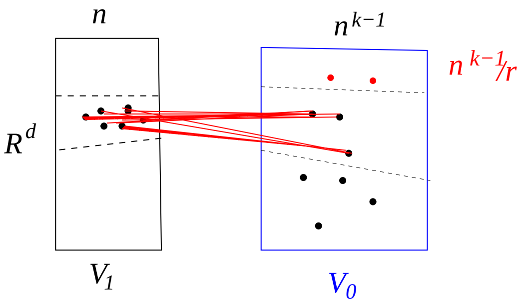
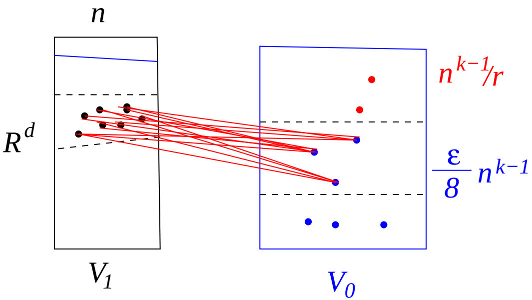
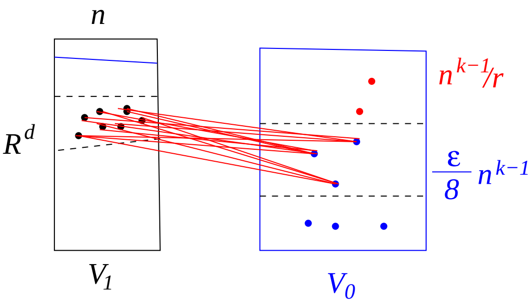


Since $|E'| \geq (\epsilon/2)n^k$ edges,

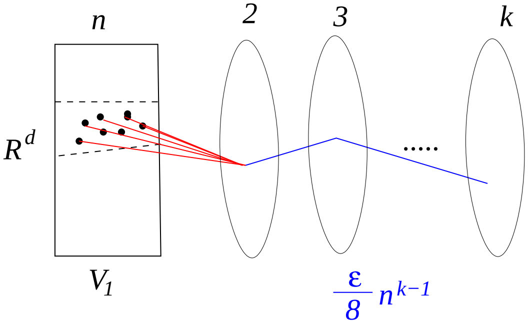
Set $r \sim 8/\epsilon$, at least $\frac{\epsilon}{8}n^{k-1}$ vertices in V_0 adjacent to all vertices in Δ.



For $r \sim 8/\epsilon$. At least $(\epsilon/2)\frac{n}{2^{10d \log d} r^d} = \frac{\epsilon^{d+1}}{2^{cd \log d}} n$ points inside Δ.



H' $(k-1)$-partite $(k-1)$-uniform hypergraphs with density $\epsilon/8$.
Apply induction hypothesis.



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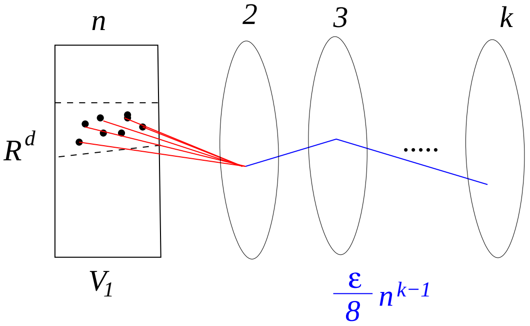
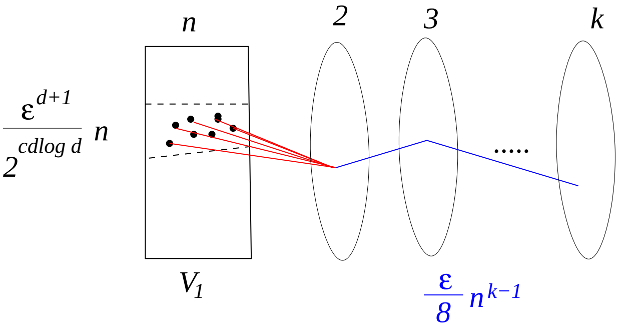


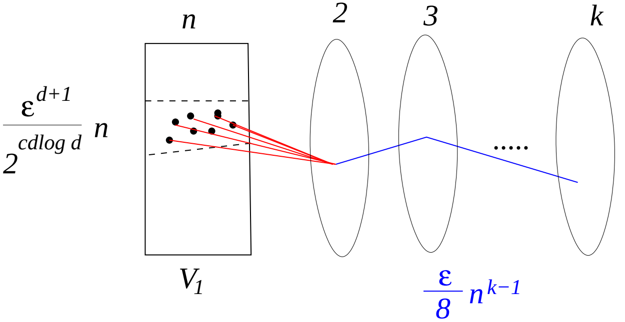
E' are $k-1$ tuples adjacent to all vertices in Δ, and that gives rise to a hyperplane $f(R^d, p) = 0$ that does not cross Δ.

H' $(k-1)$-partite $(k-1)$-uniform hypergraphs with density $\epsilon/8$.
Apply induction hypothesis.



$E' \to f(p, \mathbb{R}^{(k-1)d}) \geq 0$, and if hyperplane $f(\mathbb{R}^d, p_2, \ldots, p_k) = 0$ crosses Δ.

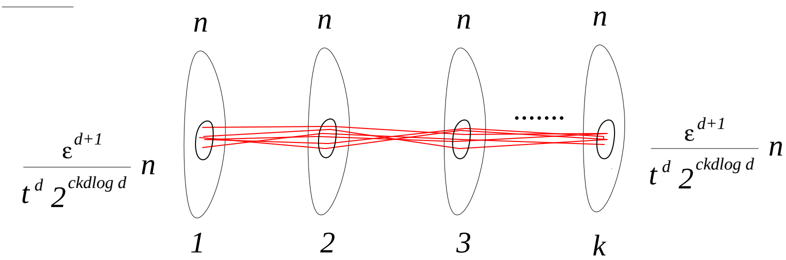
H' $(k-1)$-partite $(k-1)$-uniform hypergraphs with density $\epsilon/8$.
Apply induction hypothesis.



Find desired parts $V_1', \ldots, V_k'$

$$|V_i'| \geq \frac{\epsilon^{d+1}}{2^{ckd\log d}} n$$

Found a complete k-partite k-uniform hypergraph.



Find desired parts $V_1', \ldots, V_k'$

$$|V_i'| \geq \frac{\epsilon^{d+1}}{2^{ckd \log d}} n$$

Regularity lemma: $H = (P, E)$ semi-algebraic k-uniform hypergraph in $\mathbb{R}^d$.

Theorem (Fox, Pach, Suk, 2013+)

For any $\epsilon > 0$, we can partition P into at most $M(\epsilon)$ parts, such that almost all k-tuples of parts are **complete or empty***. Moreover $M(\epsilon) < (1/\epsilon)^c$, where c depends only on k, d, E.*

Usual regularity: almost all k-tuples of parts are "**random**". $M(\epsilon)$ is huge:

- $k = 2$, $M(\epsilon) \leq tower(1/\epsilon) = 2^{2^{2^{\cdots 2}}}$
- $k = 3$, $M(\epsilon) \leq wowzer(1/\epsilon) = tower(tower(...(tower(2))))$
- $k = 4$, $M(\epsilon) \leq wowzer(wowzer(...(wowzer(2))))$.

Future work and open problems

1. Find more applications.
2. Extend results to more complicated relations, i.e., Semi-Pfaffian, o-minimal, etc.

Thank you!