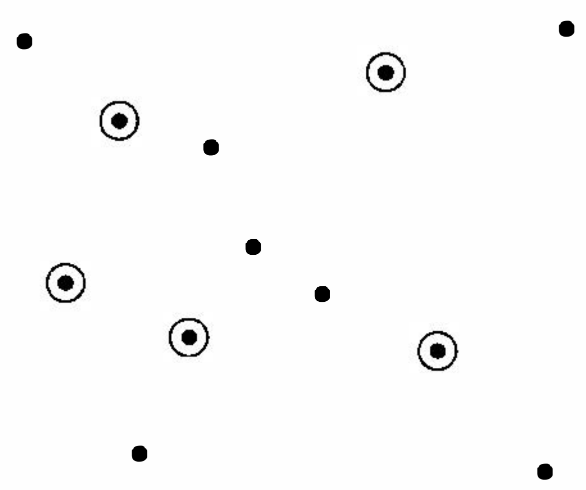
3

Convex Independent Subsets

Here we consider geometric Ramsey-type results about finite point sets in the plane. Ramsey-type theorems are generally statements of the following type: Every sufficiently large structure of a given type contains a "regular" substructure of a prescribed size. In the forthcoming Erdős–Szekeres theorem (Theorem 3.1.3), the "structure of a given type" is simply a finite set of points in general position in $\mathbf{R}^2$, and the "regular substructure" is a set of points forming the vertex set of a convex polygon, as is indicated in the picture:



A prototype of Ramsey-type results is Ramsey's theorem itself: *For every choice of natural numbers p, r, n, there exists a natural number N such that whenever X is an N-element set and $c: \binom{X}{p} \to \{1, 2, \ldots, r\}$ is an arbitrary coloring of the system of all p-element subsets of X by r colors, then there is an n-element subset $Y \subseteq X$ such that all the p-tuples in $\binom{Y}{p}$ have the same color.* The most famous special case is with $p = r = 2$, where $\binom{X}{2}$ is interpreted as the edge set of the complete graph K_N on N vertices. Ramsey's theorem asserts that if each of the edges of K_N is colored red or blue, we can always find a complete subgraph on n vertices with all edges red or all edges blue.

Many of the geometric Ramsey-type theorems, including the Erdős–Szekeres theorem, can be derived from Ramsey's theorem. But the quantitative bound for the N in Ramsey's theorem is very large, and consequently, the size of the "regular" configurations guaranteed by proofs via Ramsey's theorem is very small. Other proofs tailored to the particular problems and using more of their geometric structure often yield much better quantitative results.

3.1 The Erdős–Szekeres Theorem

3.1.1 Definition (Convex independent set). *We say that a set $X \subseteq \mathbf{R}^d$ is* convex independent *if for every $x \in X$, we have $x \notin \operatorname{conv}(X \setminus \{x\})$.*

The phrase "in convex position" is sometimes used synonymously with "convex independent." In the plane, a finite convex independent set is the set of vertices of a convex polygon. We will discuss results concerning the occurrence of convex independent subsets in sufficiently large point sets. Here is a simple example of such a statement.

3.1.2 Proposition. *Among any 5 points in the plane in general position (no 3 collinear), we can find 4 points forming a convex independent set.*

Proof. If the convex hull has 4 or 5 vertices, we are done. Otherwise, we have a triangle with two points inside, and the two interior points together with one of the sides of the triangle define a convex quadrilateral. □

Next, we prove a general result.

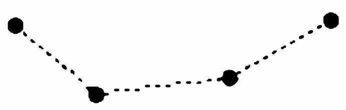
3.1.3 Theorem (Erdős–Szekeres theorem). *For every natural number k there exists a number $n(k)$ such that any $n(k)$-point set $X \subset \mathbf{R}^2$ in general position contains a k-point convex independent subset.*

First proof (using Ramsey's theorem and Proposition 3.1.2). Color a 4-tuple $T \subset X$ red if its four points are convex independent and blue otherwise. If n is sufficiently large, Ramsey's theorem provides a k-point subset $Y \subset X$ such that all 4-tuples from Y have the same color. But for $k \geq 5$ this color cannot be blue, because any 5 points determine at least one red 4-tuple. Consequently, Y is convex independent, since every 4 of its points are (Carathéodory's theorem). □

Next, we give an inductive proof; it yields an almost tight bound for $n(k)$.

Second proof of the Erdős–Szekeres theorem. In this proof, by a set in general position we mean a set with no 3 points on a common line and no 2 points having the same x-coordinate. The latter can always be achieved by rotating the coordinate system.

Let X be a finite point set in the plane in general position. We call X a *cup* if X is convex independent and its convex hull is bounded from above by a single edge (in other words, if the points of X lie on the graph of a convex function).



Similarly, we define a *cap*, with a single edge bounding the convex hull from below.

A k-cap is a cap with k points, and similarly for an ℓ-cup.

We define $f(k,\ell)$ as the smallest number N such than any N-point set in general position contains a k-cup or an ℓ-cap. By induction on k and ℓ, we prove the following formula for $f(k,\ell)$:

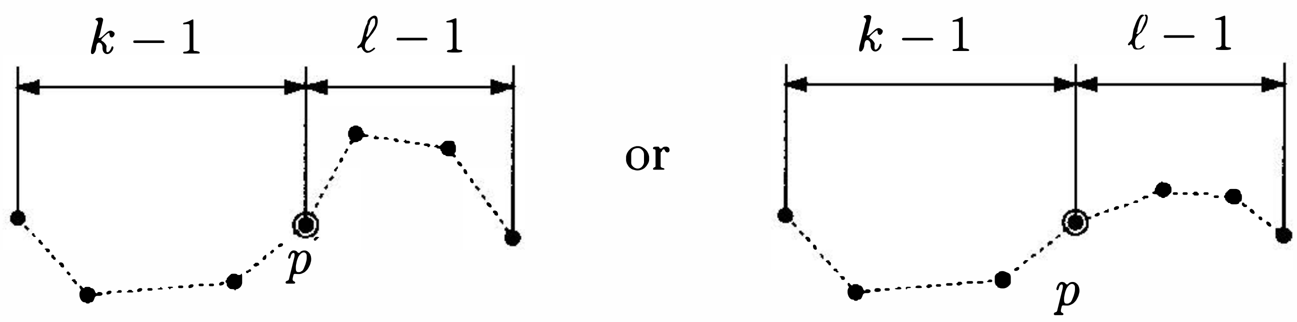
$$f(k,\ell) \leq \binom{k+\ell-4}{k-2} + 1. \tag{3.1}$$

Theorem 3.1.3 clearly follows from this, with $n(k) \leq f(k,k)$. For $k \leq 2$ or $\ell \leq 2$ the formula holds. Thus, let $k,\ell \geq 3$, and consider a set P in general position with $N = f(k-1,\ell) + f(k,\ell-1) - 1$ points. We prove that it contains a k-cup or an ℓ-cap. This will establish the inequality $f(k,\ell) \leq f(k-1,\ell) + f(k,\ell-1) - 1$, and then (3.1) follows by induction; we leave the simple manipulation of binomial coefficients to the reader.

Suppose that there is no ℓ-cap in X. Let $E \subseteq X$ be the set of points $p \in X$ such that X contains a $(k-1)$-cup ending with p.

We have $|E| \geq N - f(k-1,\ell) + 1 = f(k,\ell-1)$, because $X \setminus E$ contains no $(k-1)$-cup and so $|X \setminus E| < f(k-1,\ell)$.

Either the set E contains a k-cup, and then we are done, or there is an $(\ell-1)$-cap. The first point p of such an $(\ell-1)$-cap is, by the definition of E, the last point of some $(k-1)$-cup in X, and in this situation, either the cup or the cap can be extended by one point:

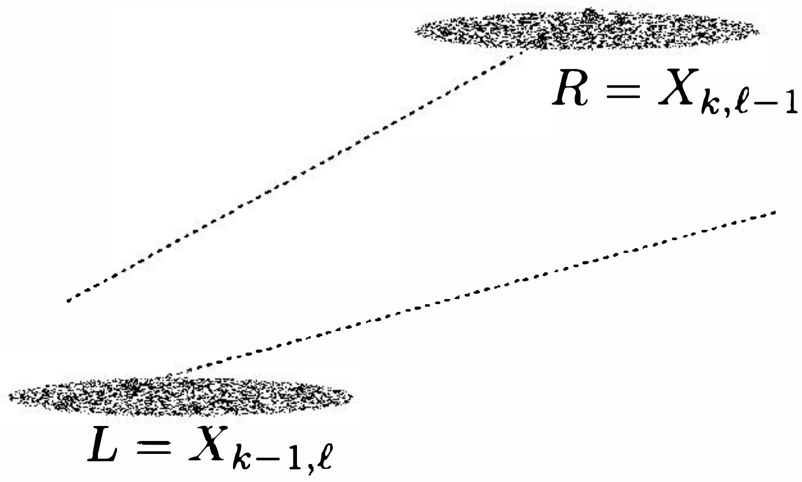


This finishes the inductive step. □

A lower bound for sets without k-cups and ℓ-caps. Interestingly, the bound for $f(k,\ell)$ proved above is tight, not only asymptotically but exactly! This means, in particular, that there are n-point planar sets in general position where any convex independent subset has at most $O(\log n)$ points, which is somewhat surprising at first sight.

An example of a set $X_{k,\ell}$ of $\binom{k+\ell-4}{k-2}$ points in general position with no k-cup and no ℓ-cap can be constructed, again by induction on $k+\ell$. If $k \leq 2$ or $\ell \leq 2$, then $X_{k,\ell}$ can be taken as a one-point set.

Supposing both $k \geq 3$ and $\ell \geq 3$, the set $X_{k,\ell}$ is obtained from the sets $L = X_{k-1,\ell}$ and $R = X_{k,\ell-1}$ according to the following picture:



The set L is placed to the left of R in such a way that all lines determined by pairs of points in L go below R and all lines determined by pairs of points of R go above L.

Consider a cup C in the set $X_{k,\ell}$ thus constructed. If $C \cap L = \emptyset$, then $|C| \leq k-1$ by the assumption on R. If $C \cap L \neq \emptyset$, then C has at most 1 point in R, and since no cup in L has more than $k-2$ points, we get $|C| \leq k-1$ as well. The argument for caps is symmetric.

We have $|X_{k,\ell}| = |X_{k-1,\ell}| + |X_{k,\ell-1}|$, and the formula for $|X_{k,\ell}|$ follows by induction; the calculation is almost the same as in the previous proof. □

Determining the exact value of $n(k)$ in the Erdős–Szekeres theorem is much more challenging. Here are the best known bounds:

$$2^{k-2} + 1 \leq n(k) \leq \binom{2k-5}{k-2} + 2.$$

The upper bound is a small improvement over the bound $f(k,k)$ derived above; see Exercise 5. The lower bound results from an inductive construction slightly more complicated than that of $X_{k,\ell}$.

Bibliography and remarks. A recent survey of the topics discussed in the present chapter is Morris and Soltan [MS00].

The Erdős–Szekeres theorem was one of the first Ramsey-type results [ES35], and Erdős and Szekeres independently rediscovered the general Ramsey's theorem at that occasion. Still another proof, also using Ramsey's theorem, was noted by Tarsi: Let the points of X be numbered $x_1, x_2, \ldots, x_n$, and color the triple $\{x_i, x_j, x_k\}$, $i < j < k$, red if we make a right turn when going from x_i to x_k via x_j, and blue if we make a left turn. It is not difficult to check that a homogeneous subset, with all triples having the same color, is in convex position.