Edge intersection graphs of systems of grid paths with bounded number of bends

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Abstract

We answer some of the questions raised by Golumbic, Lipshteyn and Stern regarding edge intersection graphs of paths on a grid (EPG graphs). We prove that for any $d \geq 4$, in order to represent all n vertex graphs with maximum degree d as edge intersection graphs of n paths, a grid of area $\Theta(n^2)$ is needed. A *bend* is a turn of a path at a grid point. Let B_k be the class of graphs that have an EPG representation such that each path has at most k bends. We show several results related to the classes B_k; among them we prove that for any odd integer k, $B_k \subsetneqq B_{k+1}$. Lastly, we show that only a very small fraction of all the $2^{\binom{n}{2}}$ labeled graphs on n vertices is in B_k.

Key words: Edge intersection graph; graph drawing.

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1 Introduction

In a recent paper [5], Golumbic, Lipshteyn and Stern introduced a notion of edge intersection graph of paths on a grid (EPG graphs) and studied some of their properties. Their research was motivated by studying graphs that come from circuit layout on a grid; see [5] for details, and [3,9] for more circuit layout problems.

Consider a family $\mathcal{F}$ of paths on a grid. The edge intersection graph of $\mathcal{F}$ is the graph whose vertices correspond to the paths from $\mathcal{F}$, two vertices being connected if and only if the coresponding paths share an edge of the grid (see Fig. 1 for an example). An *edge intersection graph of paths on a grid* (an EPG graph) is a graph which may be represented in this way. Thus EPG graphs generalize edge intersection graphs of paths on a tree (more specifically, on a tree of degree 4, studied in [6]).

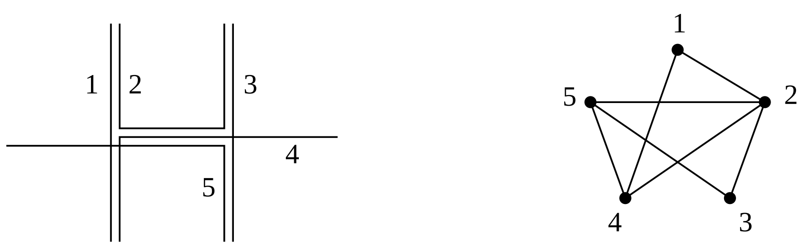

Fig. 1. An EPG realization of a graph

A *bend* is a turn of a path at a grid point. Denote by B_k the class of graphs that have an EPG representation such that each path has at most k bends. Golumbic *et al.* [5] studied these families, especially B_1. This research was motivated by the fact that in chip manufacturing, bends result in increasing the costs. In particular, they showed:

Theorem 1 *(Golumbic, Lipshteyn and Stern [5]:)*

(1) Every graph with n vertices has an EPG representation on an $n \times 2n$ grid.

(2) For each k, $B_k \subseteq B_{k+1}$; strong inclusions are conjectured.

(3) Each tree is B_1. (Follows $B_0 \subsetneqq B_1$ because not every tree is an interval graph.)

(4) $K_{3,3}$ and $K_{3,3} \setminus \{e\}$ (i.e. $K_{3,3}$ with one edge deleted) are not B_1.

(5) For any n, the complete bipartite graph $K_{m,n}$ is B_{2m-2}, see Fig. 2.

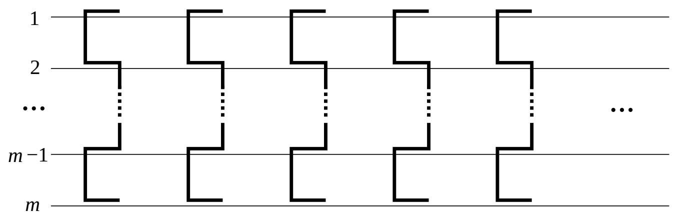

Fig. 2. $K_{m,\infty}$ is B_{2m-2}

In the present paper, we shall refine some of these results and answer some questions from [5].

Let $f_d(n)$ denote the minimum grid area required to represent all graphs on n vertices with maximum degree d. Recall (see Theorem 1(1)) that all n vertex graphs can be represented in an $n \times 2n$ grid. Since $K_{n/2,n/2}$ is a triangle free graph with $n^2/4$ edges, we need a grid of area $\Omega(n^2)$ in order to represent it as an EPG. Hence $f_{n-1}(n) = \Theta(n^2)$. However, in Section 2 we shall show that even for small d, a grid of size $\Theta(n^2)$ is required:

Theorem 2 *For fixed $d \geq 4$ and sufficiently large n, $f_d(n) = \Theta(n^2)$.*

In Section 3 we show several results concerning the classes B_k:

Theorem 3

(1) $K_{2,m}$ is B_1 if and only if $m \leq 4$. (In particular, $K_{2,5}$ is not B_1.)

(2) For fixed m and sufficiently large n, $K_{m,n}$ is not B_{2m-3} (for shortness, we write this as "$K_{m,\infty}$ is not B_{2m-3}").

(3) For an odd k, we have $B_k \subsetneq B_{k+1}$.

(4) $K_{m,n}$ is $B_{\max\{\lceil m/2\rceil,\lceil n/2\rceil\}}$. In particular, $K_{m,m}$ is $B_{\lceil m/2\rceil}$.

In Section 4 we prove that only a very small fraction of all the $2^{\binom{n}{2}}$ labeled graphs on n vertices is in B_k:

Theorem 4 *The number of labeled graphs on n vertices which can be represented as B_k-EPG is $2^{O(kn\log(kn))}$.*

2 The area of the grid

In this section, we will prove Theorem 2. Clearly $f_d(n) \leq 4n^2$, since all n vertex graphs can be represented in an $n \times 2n$ grid. For the lower bound, we will use a triangle free *expander* graph.

Observation: For fixed $d \geq 4$ and sufficiently large n, there exists a triangle free graph G with n vertices and maximum degree at most d, such that for every $S \subset V(G)$ with $|S| \leq n/2$, we have $\Gamma(S) \geq (2/3)|S| - (d-1)^3/3$ (we denote by $\Gamma(S)$ the set of vertices in $V(G) \setminus S$ that have neighbors in S).

Proof. Assume $d \geq 4$ and is even. If d is odd, then just set $d \leftarrow d-1$. Now generate a random d-regular graph on n vertices G by taking $d/2$ permutations of $V = \{1, 2, ..., n\}$, $\pi_1, ..., \pi_{d/2}$, with each π_i chosen uniformly among all $n!$ permutations and with all π_i indepedent. Then we have

$$E(G) = \{(i, \pi_j(i)), (i, \pi_j^{-1}(i)) : j = 1, ..., d/2, \quad i = 1, ..., n\}$$

Now the eigenvalues of the adjacency matrix of an undirected graph G are real and can be ordered

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$$

Since G is d-regular, $\lambda_1 = d$. Friedman [7] showed that for fixed d and sufficiently large n, $\lambda_2(G) > 2\sqrt{d-1} + .1$ holds with probability less than $1/10$. Now let X denote the number of triangles in G. Bollobas [2] showed that

$$\mathbb{E}[X] = \frac{(d-1)^3}{6}(1 + o(1)).$$

Using Markov Inequality, we obtain

$$\mathbb{P}[(\lambda_2(G) > 2\sqrt{d-1} + .1) \cup (X > (d-1)^3/3)] \leq$$

$$\leq \mathbb{P}[\lambda_2(G) > 2\sqrt{d-1} + .1] + \mathbb{P}[X > (d-1)^3/3] \leq \frac{1}{10} + \frac{\mathbb{E}[X]}{(d-1)^3/3} \leq \frac{1}{10} + \frac{1}{2} < 1.$$

Hence there exists a d-regular graph with n vertices such that $\lambda_2(G) \leq 2\sqrt{d-1} + .1$, and the number of triangles in G is less than $(d-1)^3/3$. By Alon and Milman [1], for every $S \subset V(G)$ such that $|S| \leq n/2$, we have

$$\Gamma(S) \geq \frac{2(d - (2\sqrt{d-1} + .1))}{3d - 2(2\sqrt{d-1} + .1)}|S| \geq \frac{1}{3}|S|.$$

Now we can delete at most $(d-1)^3/3$ edges in order to remove all the triangles in G, and then remove all multiple edges and loops. Hence we have a triangle free graph on n vertices and maximum degree d, such that for every $S \subset V(G)$ such that $|S| \leq n/2$, $\Gamma(S) \geq (1/3)|S| - (d-1)^3/3$. $\square$

Proof of Theorem 2. Let G be the n vertex triangle free expander graph described in the observation above. Then let $\mathcal{P}$ be a collection of n paths in an $s \times t$ grid such that the edge intersection graph of $\mathcal{P}$ is G. We claim that there exists a vertical line l which intersects at least $n/10$ paths. Indeed, assume there is no such line, and let $R(l)$ denote the paths which lay completely right to l and not intersecting l, and denote by $I(l)$ the set of paths intersecting l. By assumption, we can move the line l such that $(1/2-1/10)n < |R(l)| \leq n/2$. By construction of G, this implies for sufficiently large n

$$|I(l)| > (1/3)(1/2 - 1/10)n - \frac{(d-1)^3}{3} \geq n/10.$$

Hence we have a contradiction. Since G is triangle free, this implies

$$2t \geq |I(l)| \geq n/10 \quad \Rightarrow \quad t \geq n/20.$$

By the same argument, $s \geq n/20$, which implies

$$\frac{n^2}{400} \leq f_d(n).$$

$\square$

3 Graphs realizable as B_k-EPG

Proof of Theorem 3(1). Consider a realization of $K_{2,m}$ as B_1. Denote the paths corresponding to the "left" side of $K_{2,m}$ by a_1, a_2, those corresponding to the "right" side by $b_1, b_2, \ldots, b_m$. Denote the point of bend of a_i by A_i. Consider the following cases:

(1) A_1 and A_2 coincide;

(2) A_1 and A_2 do not coincide but lay on the same horizontal or vertical line;

(3) A_1 and A_2 do not lay on the same horizontal or vertical line.

In Case 1, each b_i passes through $A = A_1 = A_2$ and contains points of a segment of each a_i in the neighborhood of A. Therefore it is possible to add at most two b_is so that they do not meet.

In Case 2, each b_i has a horizontal segment passing through A, the midpoint of A_1A_2. Therefore it is possible to add at most one b_i.

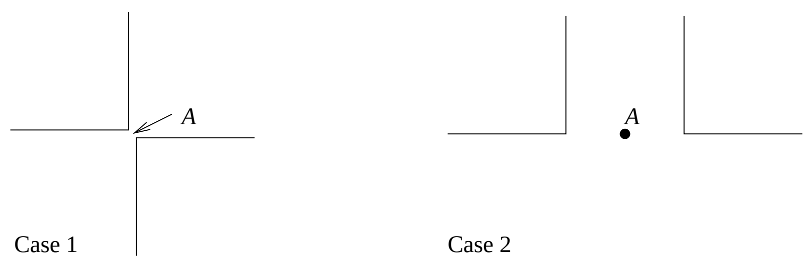

Fig. 3. Cases 1 and 2 in the proof of Theorem 3(1)

In Case 3, denote by D_1 the point of intersection of the horizontal line containing A_1 and the vertical line containing A_2, and denote by D_2 the point of intersection of the vertical line containing A_1 and the horizontal line containing A_2. The paths a_1 and a_2 may form several configurations – three of them appear on Fig. 4. However, it is easy to see that in all cases each b_i has the bend at D_1 or at D_2. Therefore it is possible to add at most four b_is: at most two having the bend at D_1, and at most two having the bend at D_2.

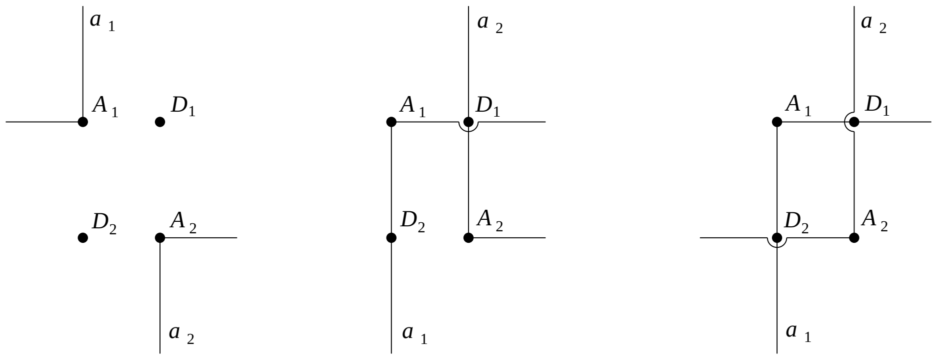

Fig. 4. Case 3 in the proof of Theorem 3(1)

Therefore for $m \geq 5$, the graph $K_{2,m}$ is not B_1. A realization of $K_{2,4}$ as B_1 is shown at Fig. 5. □

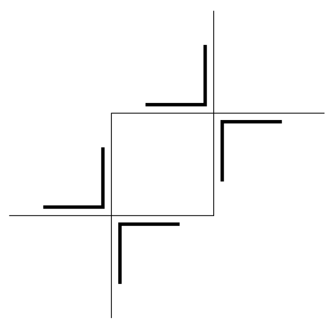

Fig. 5. $K_{2,4}$ as B_1

Proof of Theorem 3(2) and 3(3). Let m be fixed, and let n be a very large number. Consider a [hypothetic] realization of $K_{m,n}$ as B_{2m-3}. Let $a_1, a_2, \ldots, a_m$ be the paths corresponding to the "left" side of $K_{m,n}$, $b_1, b_2, \ldots, b_n$ the paths corresponding to its "right" side.

Each a_i consists of $2m-2$ segments. For each b_j consider the set of m segments, one segment from each a_i, that meet b_j (if b_j meets several segments of an a_i, choose one of them). There are $(2m-2)^m$ sets that may be obtained in this way: one of $2m-2$ segments is chosen for each of m a_is. Therefore, there is a set $A = \{\tilde{a}_i, \tilde{a}_2, \ldots, \tilde{a}_m\}$, each $\tilde{a}_i$ being a segment of a_i, and B, a subset of $\{b_1, b_2, \ldots, b_n\}$ of size at least $\frac{n}{(2m-2)^m}$, such that all the members of B (paths) meet all the members of A (segments). For each $b_j \in B$, denote by $\tilde{b}_{ij}$ the segment of b_j that meets $\tilde{a}_i$ (if some b_j has several such segments, choose one of them).

Let L be the set of all vertical and horizontal lines that contain $\tilde{a}_i$s. There are at most m lines in this set. Let P be the set of points of intersection of lines from L. The set P contains at most $(m/2)^2$ points (on the other hand it may be empty – if all the lines in L are parallel).

Delete from B all b_js that have a point from P as an inner point (there are at most $2(m/2)^2$ such paths).

Delete from B also all such b_js that $\tilde{b}_{ij}$ is not included in the interior of $\tilde{a}_i$ for some i. (That is, we delete b_js which have a $\tilde{b}_{ij}$ with at most one endpoint inside $\tilde{a}_i$.) At most $2m$ members of B are deleted at this stage. Denote the obtained subset of B by C. Since n was assumed very large, C is not empty.

Consider a path b_j in C. For each $1 \leq i \leq m$, b_j has a segment $\tilde{b}_{ij}$ contained in an $\tilde{a}_i$. Both endpoints of $\tilde{b}_{ij}$ belong to a unique $\tilde{a}_i$. Therefore b_j has at least $2m-2$ bends.

In particular, it follows from this result that $2m-2$ in Theorem 1(5) is tight: $B_{m,\infty}$ is B_{2m-2} but not B_{2m-3}. This proves Theorem 3(3): $B_k \subsetneqq B_{k+1}$ for an odd k. This also means that there is no natural k such that each graph is B_k.

Proof of Theorem 3(4). We start with a construction that shows that $K_{m,n}$ is $B_{\max\{m,n\}-1}$. Fig. 6 presents it for $K_{6,8}$. It is easy to see that it also works when m or n is odd: If we delete, say, the rightmost "vertical" path, then each "horizontal" path has an unnecessary bend and the "last" segment can be deleted.

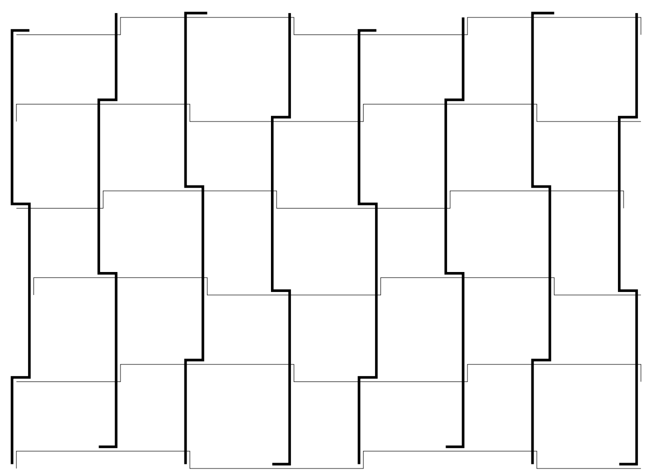

Fig. 6. $K_{6,8}$ as B_7 – an example of $K_{m,n}$ as $B_{\max\{m,n\}-1}$

However, this construction may be improved. The following construction is a realization of $K_{m,n}$ as $B_{\max\{\lceil m/2\rceil,\lceil n/2\rceil\}}$. Fig. 7 presents it for $K_{6,8}$; it is clear how to generalize it for any even m and n; if m or/and n is odd, use the construction with $m+1$ or/and $n+1$ and delete the extra path(s). $\square$

We conjecture that this construction is the best possible. In other words, that $K_{m,n}$ is not $B_{\max\{\lceil m/2\rceil,\lceil n/2\rceil\}-1}$. In particular, that $K_{m,m}$ is not $B_{\lceil m/2\rceil-1}$. If this conjecture is true, it would prove the strong inclusion $B_k \subsetneqq B_{k+1}$ for each k.

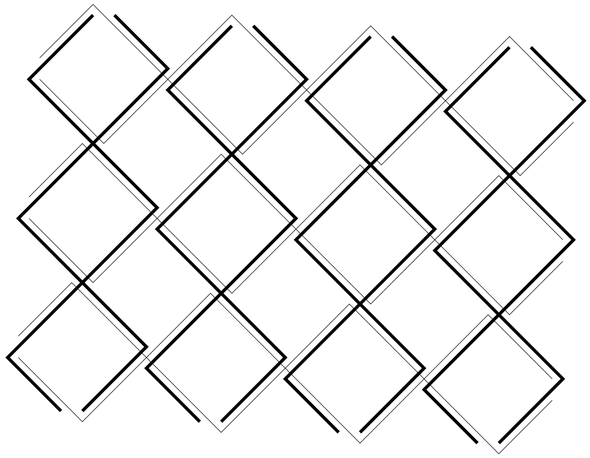

Fig. 7. $K_{6,8}$ as B_4 – an example of $K_{m,n}$ as $B_{\max\{\lceil m/2\rceil,\lceil n/2\rceil\}}$

4 Bounding the number of n vertex graphs in B_k

Proof of Theorem 4. Since each path has at most k bends, then for $r = \lfloor (k/2)+1 \rfloor$, each path consists of at most r horizontal and r vertical segments. Each path p_i can be described by the relations

$$\begin{array}{ll} x + y_{i,1}, \;\; c_{i,1} \le x \le d_{i,1} & x_{i,1} + y, \;\; c_{i,r+1} \le y \le d_{i,r+1} \\ x + y_{i,2}, \;\; c_{i,2} \le x \le d_{i,2} & x_{i,2} + y, \;\; c_{i,r+2} \le y \le d_{i,r+2} \\ \vdots \qquad\qquad \vdots & \vdots \qquad\qquad \vdots \\ x + y_{i,r}, \;\; c_{i,r} \le x \le d_{i,r} & x_{i,2r} + y, \;\; c_{i,2r} \le y \le d_{i,2r} \end{array}$$

We will assume that there are exactly r equations that represents the horizontal segments of p_i and r equations that represents the vertical lines of p_i. If there are fewer than r horizontal (or vertical) segments, we can just repeat any of the relations. Whether paths p_i and p_j cross, depends on the signs of the polynomials

$$P_{i,j,s,t} = x_{i,s} - x_{j,t} \quad Q_{i,j,s,t} = y_{i,s} - y_{j,t} \quad R_{i,j,s,t} = d_{i,s} - c_{j,t} \quad S_{i,j,s,t} = d_{j,t} - c_{i,s}$$

for $s, t \in \{1, 2, ..., r\}$, and $i, j \in \{1, 2, ..., n\}$ with $i \neq j$. Hence we have $2r^2\binom{n}{2}$ polynomials over $6rn$ variables with degree 1. Now recall that the Milnor-Thom theorem says that the number of sign patterns for m polynomials of degree d in v variables is at most $(4edm/v)^v$ [8,10,11]. Hence the number of sign patterns over our $2r^2\binom{n}{2}$ polynomials over $6rn$ variables with degree 1 is at most

$$\left(\frac{4e2r^2n^2}{6kn}\right) \leq 2^{O(kn\log kn)}$$

Hence the number of edge-intersection relationships that can be defined on n k-bend-paths in a grid is $2^{O(kn\log kn)}$. $\square$

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