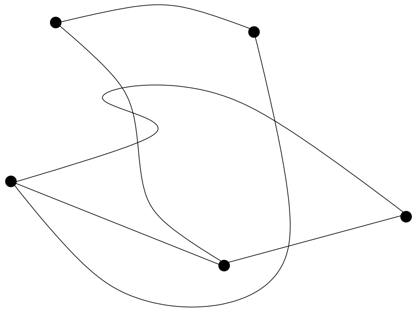
Density (Ramsey) theorems for intersection graphs of t-monotone curves

Andrew Suk

September 17, 2012

Definition

A *topological graph* is a graph drawn in the plane with vertices represented by points and edges represented by curves connecting the corresponding points. A topological graph is *simple* if every pair of its edges intersect at most once.



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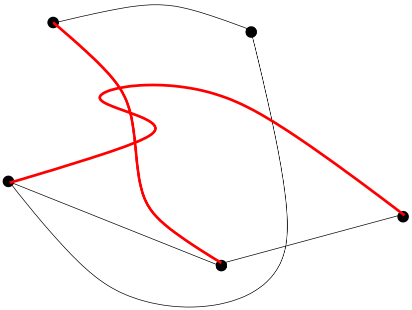
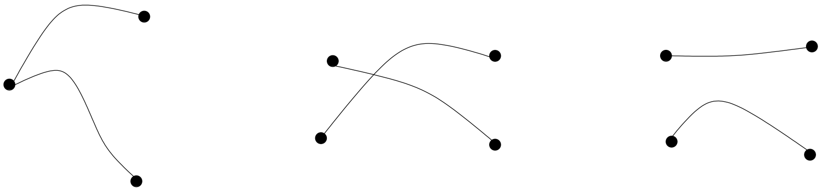


We will only consider *simple* topological graphs.



Three conjectures in topological graph theory.

conjecture 1: Thrackle conjecture.

Conjecture (Conway)

Every n-vertex simple topological graph with no two disjoint edges, has at most n edges.

Fulek and Pach 2010: $|E(G)| \leq 1.43n$.

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If edges are **segments**: Yes, Erdős.

If edges are x-**monotone**: Yes, Pach and Sterling 2011.

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conjecture 2: Extremal problem (generalization):

Conjecture (Pach and Tóth 2005, sparse graphs)

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Note: Every complete n-vertex simple topological graph has $\Omega(n^{1/3})$ pairwise disjoint edges, Suk 2011.

All solved for x-monotone curves, but all are still open for 2-monotone curves.

Conjecture (Trackle)

Every n-vertex simple topological graph with no two disjoint edges, has at most n edges.

Conjecture (Sparse problem)

Every n-vertex simple topological graph with no k pairwise disjoint edges has at most $c_k n$ edges.

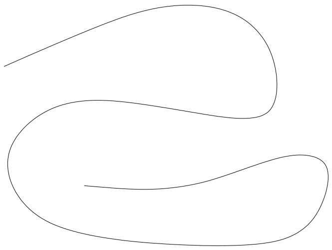
Conjecture (dense problem)

Every n-vertex simple topological graph with $\Omega(n^2)$ edges, has n^δ pairwise disjoint edges.

Definition

A curve γ is t-monotone if its interior has at most $t-1$ vertical tangent points. 1-monotone $=$ x-monotone.

Example: 4-monotone



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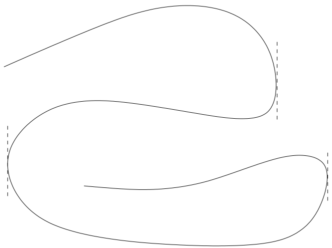


Results

Pach and Tóth's problem:

Theorem (Suk 2012)

Let G be an n-vertex simple topological graph with edges drawn as t-monotone curves. If G has no k pairwise disjoint edges, then $|E(G)| \leq n(\log n)^{c_t \log k}$.

Recall Pach and Tóth's bound of $n(\log n)^{4k-8}$ for general curves.

Results

Corollary (Suk 2012)

Let G be an n-vertex simple topological graph with edges drawn as t-monotone curves. If $|E(G)| \geq \Omega(n^2)$, then G contains $n^{\delta_t / \log \log n}$ pairwise disjoint edges.

Fox and Sudakov showed $\log^{1.02} n$ pairwise disjoint edges in the general case.

Conjecture

Every n-vertex simple topological graph with at least $\Omega(n^2)$ edges, has n^{δ} pairwise disjoint edges.

Theorem (Suk 2012)

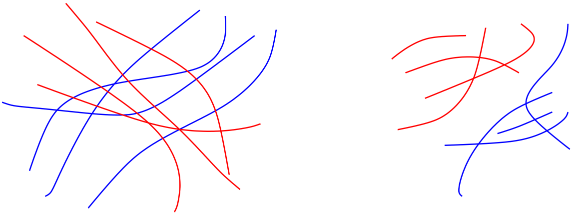
Let G be an n-vertex simple topological graph with edges drawn as t-monotone curves. If G has no k pairwise disjoint edges, then $|E(G)| \leq n(\log n)^{c_t \log k}$.

Ramsey type result

Theorem (Two color, Suk 2012)

Let R be a family of n red t-monotone curves in the plane, and let B be a family of n blue t-monotone curves in the plane, such that $R \cup B$ is simple. Then there exist subfamilies $R' \subset R$ and $B' \subset B$ such that $|R'|, |B'| \geq \epsilon n$, and either

1. *every red curve in R' intersects every blue curve in B', or*
2. *every red curve in R' is disjoint to every blue curve in B'.*



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1. For segments, Pach and Solymosi 2001.
2. Semi-algebraic sets in $\mathbb{R}^d$, Alon et al. 2005.
3. Definable sets belonging to some fixed definable family of sets in an o-minimal structure, Basu 2010.

All previous results assumed some type of bounded/fixed complexity.

Two color theorem + Szemerédi's regularity lemma ⇒ density theorem ⇒

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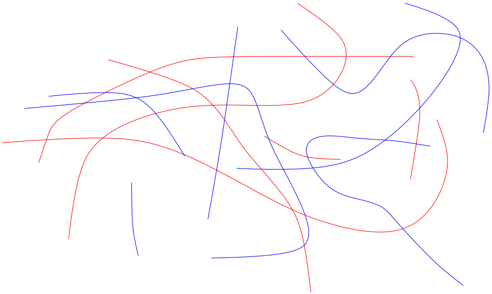

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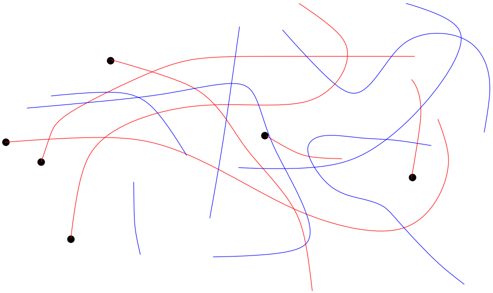

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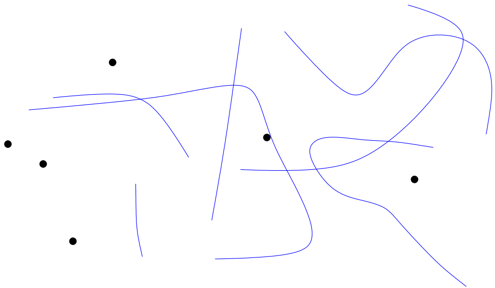
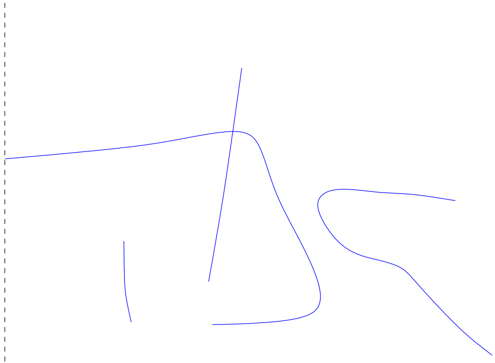


Select a random sample of c blue curves, for large constant c.

Trapezoid decomposition of $\mathbb{R}^2$: Draw a vertical line through each endpoint and through each vertical tangent point.



At most c_t^2 number of cells. With high probability, each cell will intersect at most $n/2$ blue curves!

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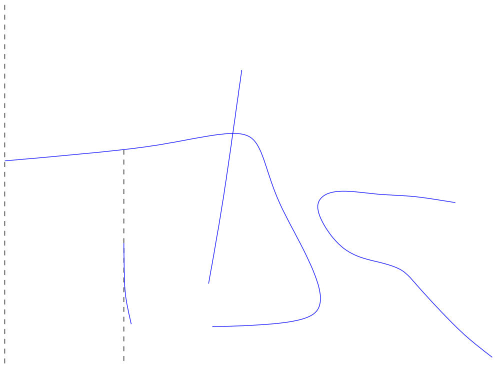

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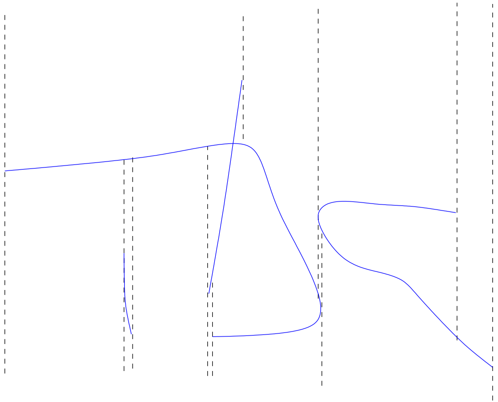
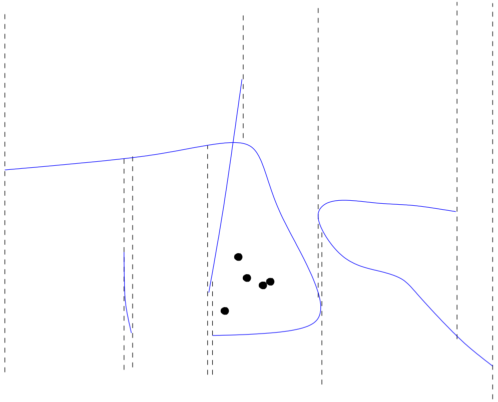
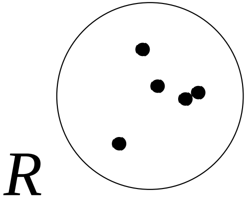


At most c_t^2 number of cells. With high probability, each cell will intersect at most $n/2$ blue curves!

By pigeonhole, there exists a cell with at least ϵn number of "left-endpoints", and $n/2$ blue curves is disjoint to this cell.

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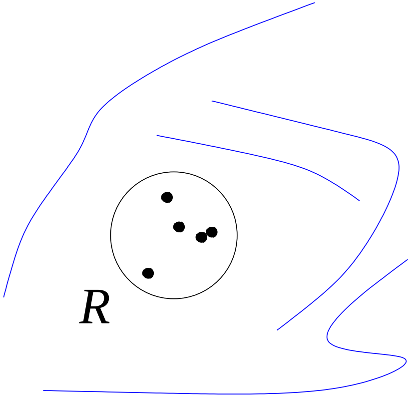
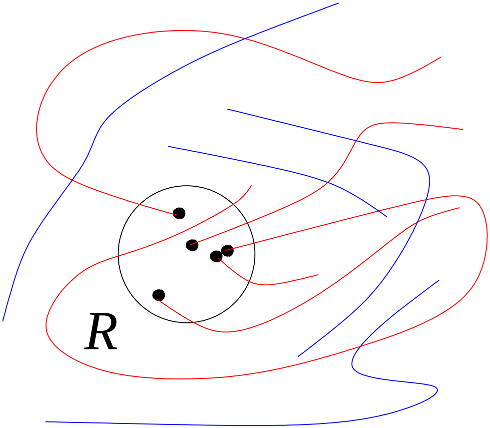
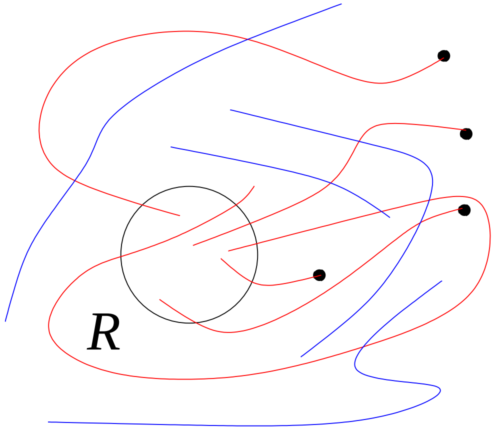

R

Look at the remaining red and blue curves.



We have ϵn red curves, and ϵn blue curves remaining.

Do it again for the remaining right-endpoints and the remaining blue curves.



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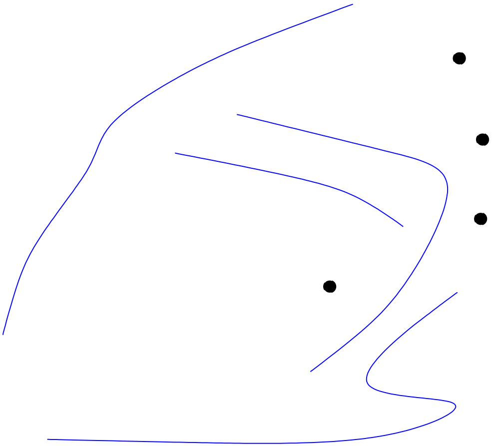
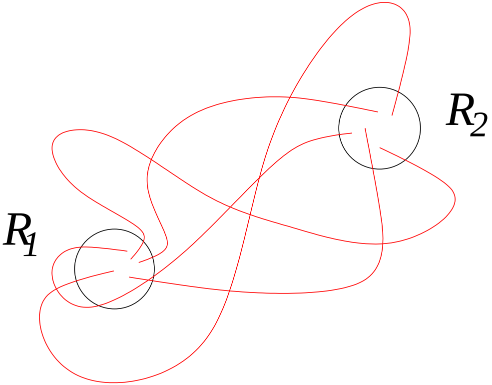
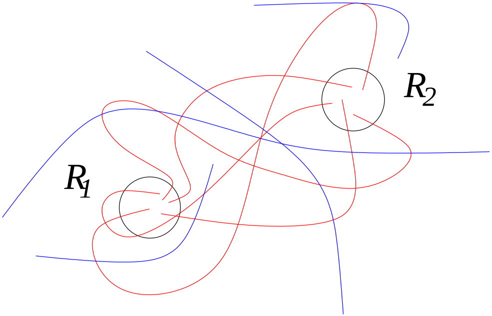


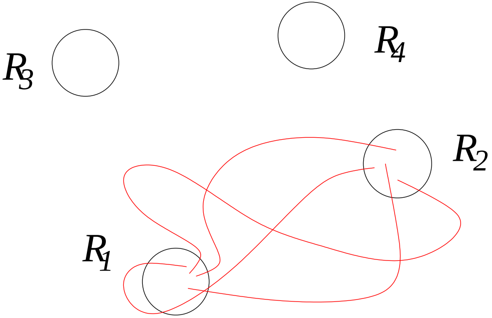
In the end, we have δn red curves, and two regions R_1 and R_2 that contains the endpoint of these red curves.



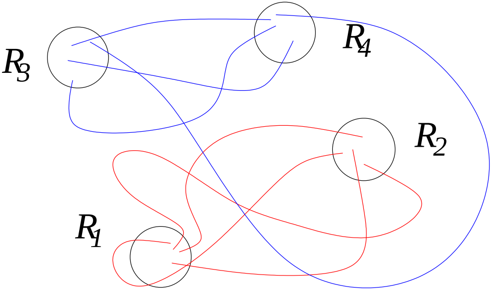
And no blue curves intersects the interior of R_1 and R_2.



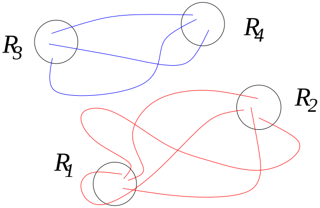
Do this whole process again with the endpoints of the blue curves to get regions R_3 and R_4.

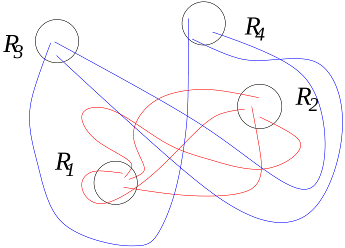


Endpoints of the blue curves lie inside R_3 and R_4, and all red curves are disjoint to the interior of R_3 and R_4.



Apply a case analysis/Jordan curve argument to find:





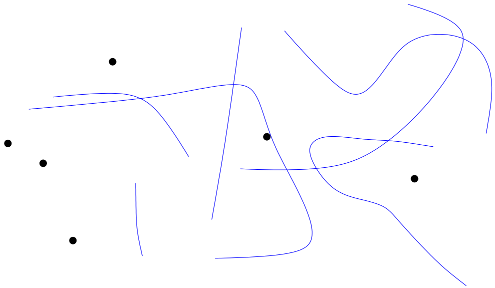
End of "proof".

Open problem

t-monotone condition only used for the trapezoid decomposition.

Problem

Given an n-point set P and family F of n simple curves, such that no point lies on any curve in F, does there exist a region R that contain ϵn points, and the interior of R is disjoint to ϵn curves from F?



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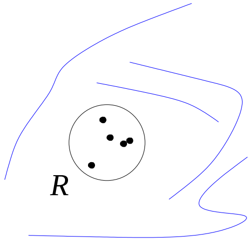

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Let G be an n-vertex simple topological graph with edges drawn as 2-monotone curves. If G does not contain 2 disjoint edges, then $|E(G)| \leq n$?

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Given a simple family F of 2-monotone curves in the plane with no 3 pairwise disjoint members, $\chi(\overline{F}) \leq c$ for some constant c?

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Note: Color problem is true for segments/x-monotone curves.

Thank you!