Approximating the rectilinear crossing number

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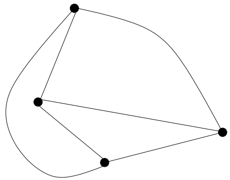
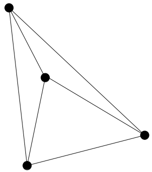
Crossing number $\mathrm{cr}(G)$ = minimum number of crossing pairs of edges over all drawings of G.

Rectilinear crossing number $\overline{\mathrm{cr}}(G)$ = minimum number of crossing pairs of edges over all *straight-line* drawings of G





cr versus $\overline{\mathrm{cr}}$

$$\mathrm{cr}(G) \leq \overline{\mathrm{cr}}(G)$$

Theorem (Fáry 1948)

$\mathrm{cr}(G) = 0$ *if and only if* $\overline{\mathrm{cr}}(G) = 0$.

Theorem (Bienstock and Dean 1993)

There is a sequence of graphs $G_1, G_2, \ldots, G_m, \ldots$ *such that*

$$\mathrm{cr}(G_m) = 4 \qquad \overline{\mathrm{cr}}(G_m) \geq m$$

Rectilinear crossing number

Computing $\overline{\text{cr}}(G)$ is NP-hard (Bienstock 1991)

Open problem: Determine the asymptotic value of $\overline{\text{cr}}(K_n)$.

Theorem (Ábrego et al. 2012 and Fabila-Monroy and López 2014)

$$0.379972\binom{n}{4} < \overline{\text{cr}}(K_n) < 0.380473\binom{n}{4}$$

Main result: approximating $\overline{\mathrm{cr}}(G)$

Theorem (Fox, Pach, S. 2016)

There is a deterministic $n^{2+o(1)}$-time algorithm for constructing a straight-line drawing of any n-vertex graph G in the plane with

$$\overline{\mathrm{cr}}(G) + \frac{cn^4}{(\log \log n)^{c'}}$$

crossing pairs of edges.

Crossing Lemma

Lemma (Ajtai, Chvátal, Newborn, Szemerédi 1982 and Leighton 1983)

Let G be a graph on n vertices and e edges. Then

$$\overline{\mathrm{cr}}(G) \geq \frac{e^3}{64n^2} - 4n$$

Dense graph: $|E(G)| \geq \epsilon n^2$, we have $\overline{\mathrm{cr}}(G) = \alpha n^4 + o(n^4)$.

A $(1 + o(1))$-approximation for dense graphs

Theorem (Fox, Pach, Suk 2016)

There is a deterministic $n^{2+o(1)}$-time algorithm for constructing a straight-line drawing of any n-vertex graph G with $|E(G)| > \epsilon n^2$, such that the drawing has at most

$$\overline{\mathrm{cr}}(G) + o(\overline{\mathrm{cr}}(G))$$

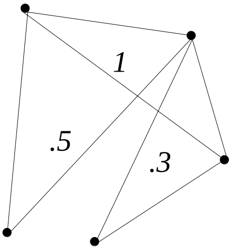
crossing pairs of edges.

Generalization to weighted graphs

Edge weighted graphs. $G = (V, E)$, $w_G : E \to [0, 1]$.

For a fixed drawing $\mathcal{D}$, the weighted number of crossings is

$$\sum_{(e,e')\in X_D} w_G(e) \cdot w_G(e')$$



Example: $0.5 + 0.3 = 0.8$

Generalization to weighted graphs

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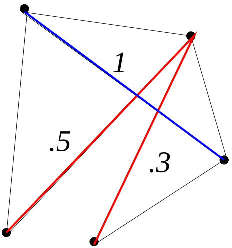

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Rectilinear crossing number of edge-weighted graphs

$$\overline{\mathrm{cr}}(G) = \min_{\mathcal{D}} \sum_{(e,e')\in X_{\mathcal{D}}} w_G(e) \cdot w_G(e')$$

If G is not weighted,

$w_G(e) = 1$ if $e \in E(G)$
$w_G(e) = 0$ if $e \notin E(G)$.

Theorem (Frieze-Kannan 1999)

For any $\epsilon > 0$*, every graph* $G = (V, E)$ *has a equitable vertex partition* $V = V_1 \cup \cdots \cup V_K$, $1/\epsilon < K < 2^{c\epsilon^{-2}}$*, such that for all disjoint subsets* $S, T \subset V(G)$

$$\left| e(S,T) - \sum_{1 \le i,j \le K} e(V_i, V_j) \frac{|S \cap V_i||T \cap V_j|}{(n/K)^2} \right| < \epsilon n^2$$

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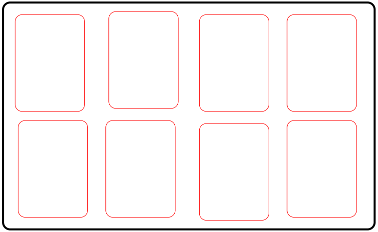
$G =$ 

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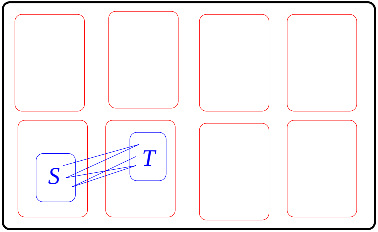
$G =$ 

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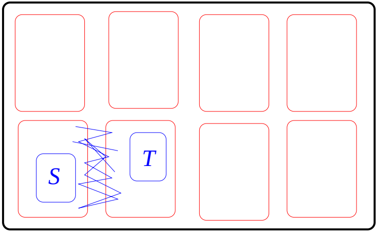
$G =$ 

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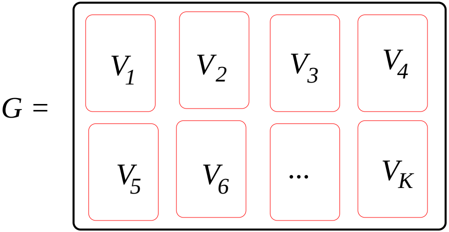
Theorem (Dellamonica et al. 2015)

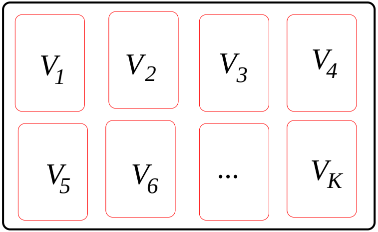
There is a determinist $2^{2^{\epsilon^{-c}}} n^2$-time algorithm for computing such a partition.

Given a Frieze-Kannan regular partition $\mathcal{P} : V = V_1 \cup \cdots \cup V_K$:



Given a Frieze-Kannan regular partition $\mathcal{P} : V = V_1 \cup \cdots \cup V_K$:

$G_P =$



$w_{G_{\mathcal{P}}}(uv) = 0$ if $u, v \in V_i$,

$w_{G_{\mathcal{P}}}(uv) = \frac{e_G(V_i, V_j)}{(n/K)^2}$ if $u \in V_i, v \in V_j$

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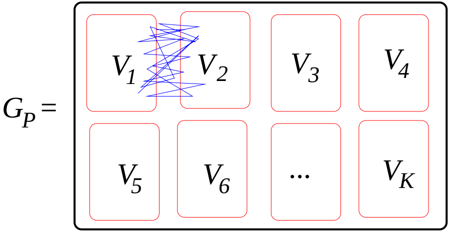

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$G \approx G_{\mathcal{P}}$: For $S, T \subset V$, $|e_G(S, T) - e_{G_{\mathcal{P}}}(S, T)| < \epsilon n^2$.

$\overline{\mathrm{cr}}(G)$ versus $\overline{\mathrm{cr}}(G_{\mathcal{P}})$

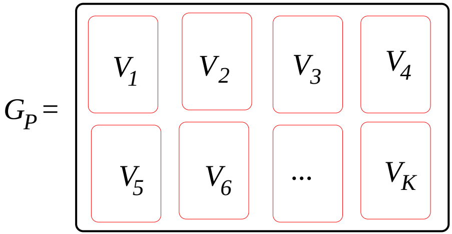
Using the regularity lemma for same-type transversals (Fox-Pach-S.) **Key Lemma**

Lemma (Fox, Pach, S. 2016)

$$|\overline{\mathrm{cr}}(G) - \overline{\mathrm{cr}}(G_{\mathcal{P}})| < \epsilon^{\frac{1}{4C}} n^4$$

Defining $G/\mathcal{P}$

Given a Frieze-Kannan regular partition $\mathcal{P} : V = V_1 \cup \cdots \cup V_K$:



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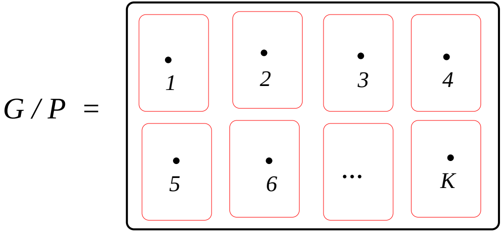

$$V(G/\mathcal{P}) = \{1, 2, \ldots, K\}$$

$$w_{G/\mathcal{P}}(ij) = \frac{e_G(V_i, V_j)}{(n/K)^2} \text{ if } u \in V_i, v \in V_j$$

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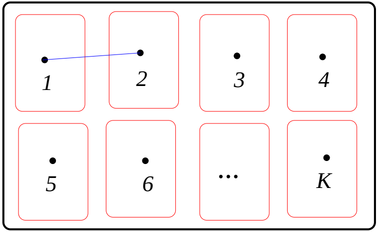
$G/P =$ 

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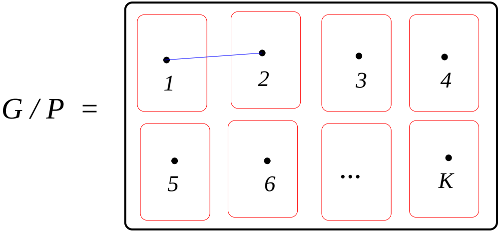
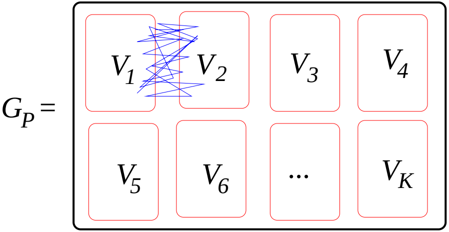




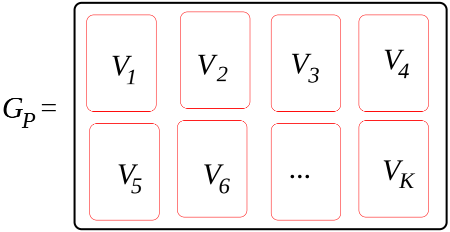
$G_{\mathcal{P}}$ is an (n/K)-blow up of $G/\mathcal{P}$.

Simple Lemma

Lemma

$$\left(\frac{n}{K}\right)^4 \overline{\mathrm{cr}}(G/P) \leq \overline{\mathrm{cr}}(G_{\mathcal{P}})$$

Proof. Consider a drawing of $G_{\mathcal{P}}$ with $\overline{\mathrm{cr}}(G_{\mathcal{P}})$ weighted crossings.



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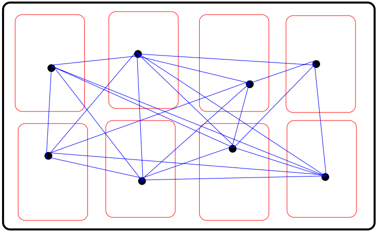

G/P

at least $\overline{\mathrm{cr}}(G/\mathcal{P})$ weighted crossings.

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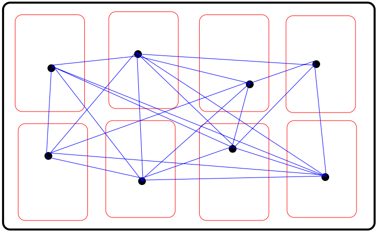
$G_P =$ 

G/P

Summing over all $(n/K)^K$ drawings gives a total of $\ge (n/K)^K \overline{\mathrm{cr}}(G/\mathcal{P})$.

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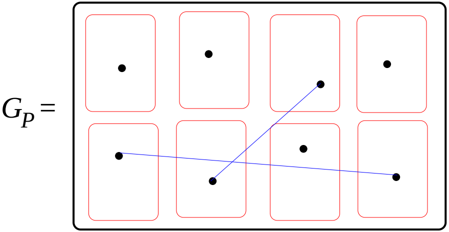

G/P

Each fixed crossing with be counted $(n/K)^{K-4}$ times, giving $(n/K)^{K-4}\overline{\mathrm{cr}}(G_{\mathcal{P}}) \geq (n/K)^K \overline{\mathrm{cr}}(G/\mathcal{P})$

The algorithm

Input: $G = (V, E)$.

1) Set $\epsilon = (\log \log n)^{-1/2c}$

2) Compute the Frieze-Kannan vertex partition on $V = V_1 \cup \cdots \cup V_K$, $K \leq 2^{\epsilon^{-c}} = 2^{\sqrt{\log \log n}}$. Done in $n^{2+o(1)}$-time.

3) Find a straight-line drawing of $G/\mathcal{P}$ with $\overline{\mathrm{cr}}(G/\mathcal{P})$ weighted pairs of crossing edges. Done in $2^{O(K^3)} = n^{o(1)}$ time.

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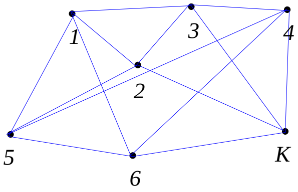

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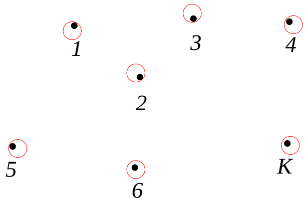
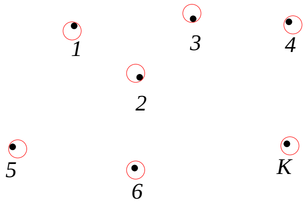


4) Place all vertices from V_i inside circle C_i. Done in $O(n)$-time.



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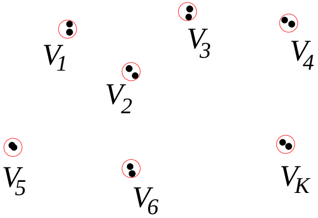
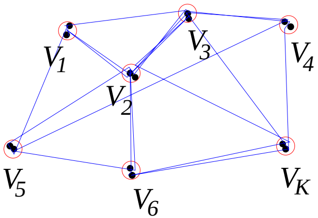


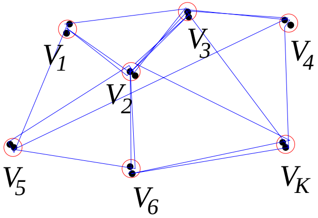
4) Place all vertices from V_i inside circle C_i. Done in $O(n)$-time.

5) Draw all remaining edges. Done in $O(n^2)$-time.



6) **Return:** drawing of G. Total running time: $O(n^{2+o(1)})$

X denote the set of pairs of crossing edges.



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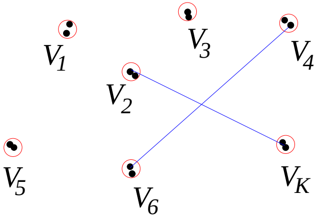


$$|X| \leq \left(\frac{n}{K}\right)^4 \overline{\mathrm{cr}}(G/P) + \frac{n^4}{2K}$$

Number of crossings in the drawing

Simple Lemma: $\left(\frac{n}{K}\right)^4 \overline{\text{cr}}(G/P) \leq \overline{\text{cr}}(G_{\mathcal{P}})$.

$$|X| \leq \left(\frac{n}{K}\right)^4 \overline{\text{cr}}(G/P) + \frac{n^4}{2K} \leq \overline{\text{cr}}(G_{\mathcal{P}}) + \frac{n^4}{2K}$$

Key Lemma: $\overline{\text{cr}}(G_{\mathcal{P}}) \leq \overline{\text{cr}}(G) + \epsilon^{1/4C} n^4$

$$|X| \leq \overline{\text{cr}}(G) + \epsilon^{1/4C} n^4 + \frac{n^4}{2K}$$

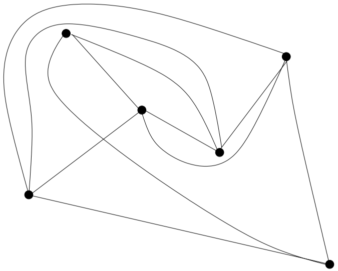
$$|X| \leq \overline{\text{cr}}(G) + \frac{n^4}{(\log \log n)^{c'}}$$

Open problem: Generalize to from $\overline{\mathrm{cr}}$ to cr

Suffices to generalize the key lemma.

Key Lemma: $\overline{\mathrm{cr}}(G_{\mathcal{P}}) \leq \overline{\mathrm{cr}}(G) + \epsilon^{1/4C} n^4$

Open problem: $\mathrm{cr}(G_{\mathcal{P}}) \leq \mathrm{cr}(G) + \epsilon^{1/4C} n^4$



Thank you!