Geometric Ramsey Theory

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Introduction

For k-uniform hypergraphs.

Definition

We define the *Ramsey number* $R_k(n)$ to be the minimum integer N such that any N-vertex k-uniform hypergraph H contains either a clique or an independent set of size n.

Theorem (Ramsey 1930)

For all k, n, the Ramsey number $R_k(n)$ is finite.

Estimate $R_k(n)$, k fixed and $n \to \infty$.

Known estimates

Theorem (Erdős-Szekeres 1935, Erdős 1947)

$$2^{n/2} \leq R_2(n) \leq 2^{2n}.$$

Theorem (Erdős-Rado 1952, Erdős-Hajnal 1960's)

$$2^{cn^2} \leq R_3(n) \leq 2^{2^{c'n}}.$$

$$t_{k-1}(cn^2) \leq R_k(n) \leq t_k(c'n).$$

Tower function $t_i(x)$ is given by $t_1(x) = x$ and $t_{i+1}(x) = 2^{t_i(x)}$.

Combinatorial Problem

$$2^{cn^2} \leq R_3(n) \leq 2^{2^{c'n}}.$$

Problem

Close the gap on $R_3(n)$

Conjecture (Erdős, $500 problem)

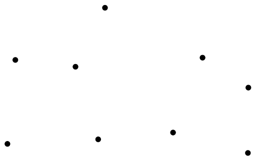
$$2^{2^{cn}} \leq R_3(n)$$

Erdős-Hajnal Stepping Up Lemma: $x < R_k(n)$, then $2^x \lesssim R_{k+1}(n)$ for $k \geq 3$

Would imply $R_4(n) = 2^{2^{2^{\Theta(n)}}}$, and $R_k(n) = t_k(\Theta(n))$.

Is there a geometric construction showing $2^{2^{cn}} \leq R_3(n)$?

$ES(n)$ exists



$V = \{N \text{ points in the plane}\}$,
$E = \{\text{triples having a clockwise orientation}\}$.

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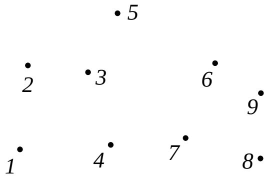

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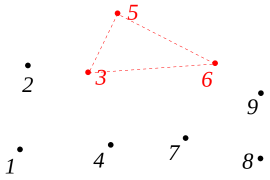

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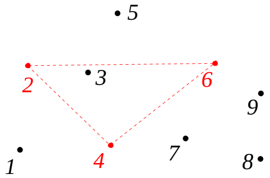

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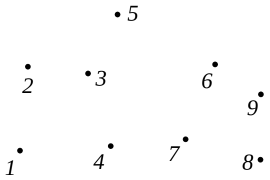
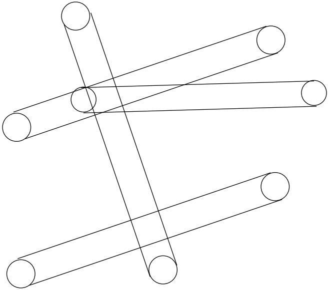


$V = \{N \text{ points in the plane}\}$,
$E = \{\text{triples having a clockwise orientation}\}$.
Many graphs and hypergraphs defined geometrically.

$V = \{N$ tubes of length l and radius 1 in $\mathbb{R}^d\}$
$E = \{$pairs that intersect$\}$.



Semi-algebraic hypergraphs.

semi-algebraic sets (Tarski cell)

Definition

A set $A \subset \mathbb{R}^d$ is called *semi-algebraic* if there are polynomials $f_1, f_2, ..., f_r \in \mathbb{R}[x_1, ..., x_d]$ and a Boolean formula $\Phi(X_1, X_2, ..., X_r)$, where $X_1, ..., X_r$ are variables attaining values "true" and "false", such that

$$A = \left\{ x \in \mathbb{R}^d : \Phi\left(f_1(x) \geq 0, ..., f_r(x) \geq 0\right) \right\}.$$

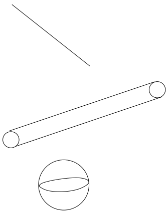
Φ involves unions, intersections, and complementations. Assume Quantifier-free (Tarski's Theorem).

A has *complexity at most* t if $d, r \leq t$ and each $deg(f_i) \leq t$.

Examples: hyperplanes, balls, boxes, tubes, etc. in $\mathbb{R}^d$.

Encode sets to points

Let $V = \{A_1, \ldots, A_N\}$ be a family of N semi-algebraic sets in $\mathbb{R}^d$, each set with complexity at most t.



$$A_i = \left\{ x \in \mathbb{R}^d : \Phi\left(f_1(x) \geq 0, \ldots, f_r(x) \geq 0\right) \right\}.$$

Encode each set: $A_i \rightarrow p_i \in \mathbb{R}^q$ for $q = q(t)$.

$V = \{p_1, \ldots, p_N\}$, N points in $\mathbb{R}^q$.

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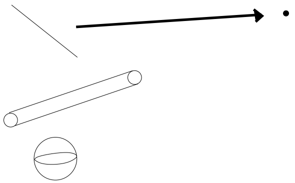

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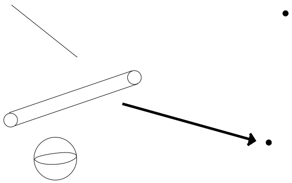

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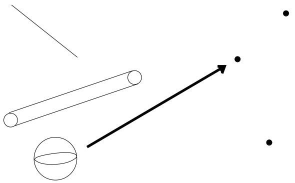

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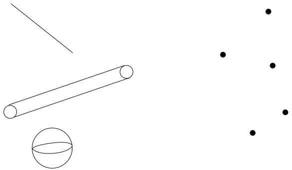

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$V = \{p_1, ..., p_N\}$, N points in $\mathbb{R}^q$.

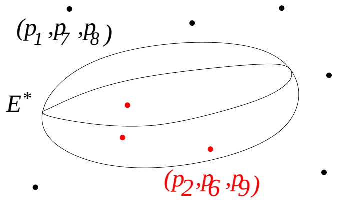
Semi-algebraic relation

For $V = \{p_1, ..., p_N\} \subset \mathbb{R}^q$, the edge set $E \subset \binom{V}{k}$ is semi-algebraic if E can be described with a constant number of polynomial equations and inequalities (each of bounded degree), and a boolean formula Φ.

Semi-algebraic relation

For $V = \{p_1, ..., p_N\} \subset \mathbb{R}^q$, the edge set $E \subset \binom{V}{k}$ is semi-algebraic if there exists a semi-algebraic set $E^* \subset \mathbb{R}^{kq}$ with bounded description complexity, such that for $i_1 < \cdots < i_k$

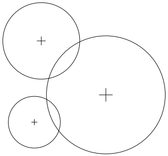
$$(p_{i_1}, ..., p_{i_k}) \in E \Leftrightarrow (p_{i_1}, ..., p_{i_k}) \in E^* \subset \mathbb{R}^{kq}.$$



Example: For $k = 3$ look at all triples $(p_{i_1}, p_{i_2}, p_{i_3})$ in $\mathbb{R}^{3q}$.

Call the pair (V, E) a semi-algebraic k-uniform hypergraph (with bounded description complexity).

$V = \{A_1, ..., A_N\}$, N disks in the plane. $E = \{$pairs of disks that intersect$\}$.



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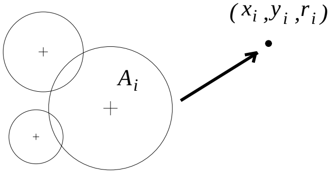

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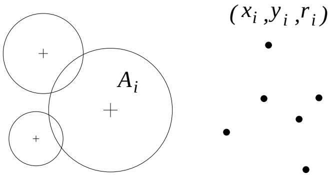

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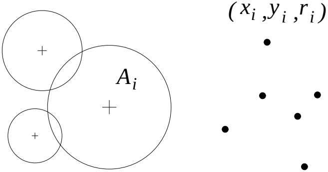
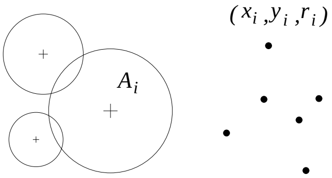


$A_i \to p_i = (x_i, y_i, r_i)$, $A_j \to p_j = (x_j, y_j, r_j)$. A_i and A_j cross if and only if

$$-x_i^2 + 2x_ix_j - x_j^2 - y_i^2 + 2y_iy_j - y_j^2 + r_i^2 + 2r_ir_j + r_j^2 \geq 0.$$

$V = \{A_1, ..., A_N\}$, N disks in the plane. $E = \{$ pairs of disks that intersect$\}$.



(V, E) is semi-algebraic graph,

$E^* = \{(z_1, ..., z_6) \in \mathbb{R}^6 : f(z_1, ..., z_6) \geq 0\}$, where

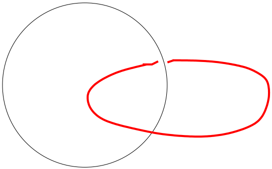
$$f(z_1, ..., z_6) = -z_1^2 + 2z_1z_4 - z_4^2 - z_2^2 + 2z_2z_5 - z_5^2 + z_3^2 + 2z_3z_6 + z_6^2.$$

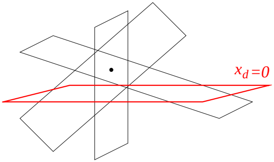
$(p_i, p_j) \in E \Leftrightarrow (p_i, p_j) \in E^*$.

Examples

1. $V = \{N \text{ circles in } \mathbb{R}^3\}$
   $E = \{\text{pairs that are linked}\}$.

2. $V = \{N \text{ hyperplanes in } \mathbb{R}^d \text{ in general position}\}$,
   $E = \{d\text{-tuples whose intersection point is above the hyperplane } x_d = 0\}$.





Results

Definition: Let $R_k^{semi}(n)$ be the minimum integer N such that any N-vertex semi-algebraic k-uniform hypergraph $H = (V, E)$ contains either a clique or an independent set of size n. $R_k^{semi}(n) \leq R_k(n)$.

Theorem (Alon, Pach, Pinchasi, Radoičić, Sharir 2005)

$$R_2^{semi}(n) \leq n^{c_1}.$$

Applying Milnor-Thom Theorem and Cutting Lemma:

Theorem (Conlon, Fox, Pach, Sudakov, S. 2012)

for $k \geq 3$,

$$t_{k-1}(c_2 n) \leq R_k^{semi}(n) \leq t_{k-1}(n^{c_1}).$$

Recall: for $k \geq 3$, $t_{k-1}(cn^2) \leq R_k(n) \leq t_k(c'n)$.

Theorem (Conlon, Fox, Pach, Sudakov, S. 2012)

for $k \geq 3$,

$$t_{k-1}(c_2 n) \leq R_k^{semi}(n) \leq t_{k-1}(n^{c_1}).$$

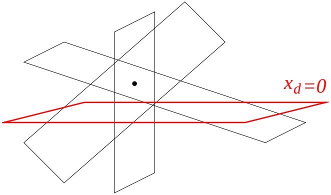
Several applications...

Problem (Matoušek-Welzl 1992, Dujmović-Langerman 2011, Matoušek-Eliáš 2012.)

Determine the minimum integer $OSH_d(n)$, such that any family of at least $OSH_d(n)$ hyperplanes in $\mathbb{R}^d$ in general position, must contain n members such that every d-tuple intersects on one-side of the hyperplane $x_d = 0$.

$$OSH_2(n) = \Theta(n^2), \qquad OSH_d(n) \leq R_d(n) \leq t_d(c'n).$$

$V = \{\text{N hyperplanes}\}$,
$E = \{d\text{-tuples that intersect above } x_d = 0 \text{ hyperplane}\}$.
New bound: $OSH_d(n) \leq R_d^{semi}(n) \leq t_{d-1}(n^{c_1})$



Combinatorial Problem

Ramsey number of 3-uniform hypergraphs.

$$2^{cn^2} \leq R_3(n) \leq 2^{2^{c'n}}.$$

Conjecture (Erdős)

$$2^{2^{cn}} \leq R_3(n)$$

Is there a geometric construction showing $2^{2^{cn}} \leq R_3(n)$?

Our Result: $R_3^{semi}(n) \leq 2^{n^{c_1}}$.

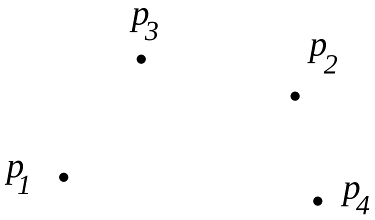
Another Ramsey-type problem in geometry.

Given a point sequence $P = p_1, p_2, ..., p_N \subset \mathbb{R}^d$ in general position, $\chi : \binom{P}{d+1} \rightarrow \{+1, -1\}$ (positive or negative orientation).

χ is the *order-type* of P.

Orientation: of a $(d+1)$-tuples $p_1, p_2, ..., p_{d+1} \in \mathbb{R}^d$ in general position.

Example: in $\mathbb{R}^3$, p_1, p_2, p_3, p_4.



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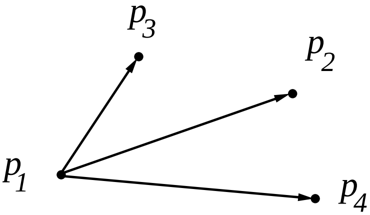

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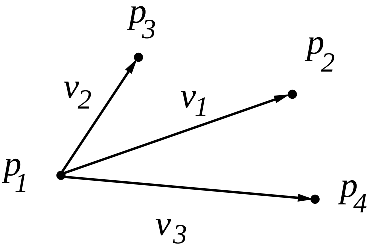

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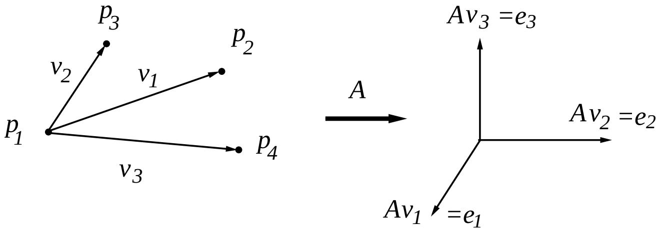
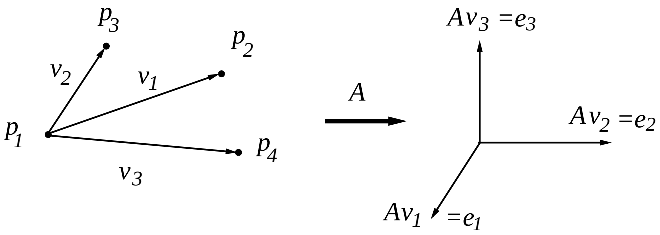


$\chi(\{p_1, p_2, p_3, p_4\}) = \operatorname{sgn}\det(A)$.

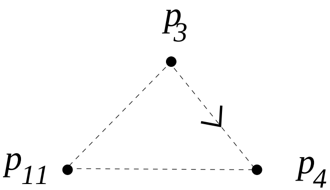
Orientation: of a $(d+1)$-tuples $p_1, p_2, ..., p_{d+1} \in \mathbb{R}^d$ in general position.

Example: in $\mathbb{R}^3$, p_1, p_2, p_3, p_4.



Which side of the hyperplane $h = (p_1, ..., p_d)$ the point p_{d+1} lies.

Example: 2-dimensions, $d + 1 = 3$



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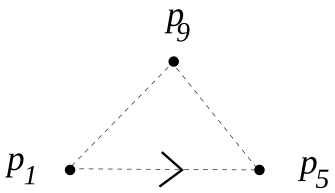
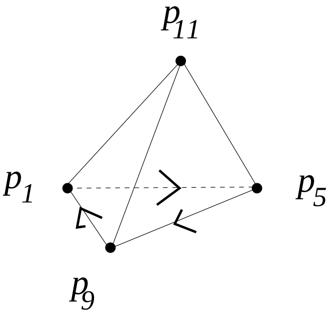


Example: 3-dimensions, $d + 1 = 4$



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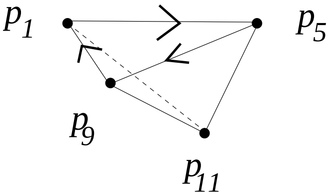

A point sequence $P = p_1, ..., p_n \subset \mathbb{R}^d$ is *order-type homogeneous*, if every $d+1$-tuple has the same orientation (i.e. all positive or all negative).

Problem (Corodovil-Duchet 2000, Matoušek-Eliáš 2012.)

Determine the minimum integer $OT_d(n)$, such that any sequence of $OT_d(n)$ points in $\mathbb{R}^d$ in general position, contains an n-element subsequence that is order-type homogeneous.

A point sequence $P = p_1, ..., p_n \subset \mathbb{R}^d$ is *order-type homogeneous*, if every $d+1$-tuple has the same orientation (i.e. all positive or all negative).

Fact

A point sequence that is order-type homogeneous forms the vertex set of a convex polytope combinatorially equivalent to the cyclic polytope in $\mathbb{R}^d$.

A point sequence $P = p_1, ..., p_n \subset \mathbb{R}^d$ is *order-type homogeneous*, if every $d+1$-tuple has the same orientation (i.e. all positive or all negative).

Theorem (McMullen 1962)

Among all d-dimensional convex polytopes with n vertices, the cyclic polytope maximizes the number of faces of each dimension

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Determine the minimum integer $OT_d(n)$, such that any sequence of $OT_d(n)$ points in $\mathbb{R}^d$ in general position, contains an n-element subsequence that is order-type homogeneous.

1-dimension: $P = p_1, ..., p_N \subset \mathbb{R}$, order-type homogeneous subset

$$p_{i_1} < p_{i_2} < \cdots < p_{i_n}$$

or

$$p_{i_1} > p_{i_2} > \cdots > p_{i_n}$$

Erdős-Szkeres: $OT_1(n) = (n-1)^2 + 1$

2-dimensions: Order-type homogeneous subset: Every triple has a clockwise orientation, or every triple has a clockwise orientation.

$OT_2(n)$ is about points in convex position.

Erdős-Szkeres cups-caps Theorem: $OT_2(n) = 2^{\Theta(n)}$

For $d \geq 3$

$V = \{N$ labeled points in $\mathbb{R}^d$ in general position$\}$
$E = \{(d+1)$-tuples having a positive orientation$\}$

- $OT_d(n) \leq R_{d+1}(n) \leq t_{d+1}(O(n))$
- $OT_d(n) \leq R_{d+1}^{semi}(n) \leq t_d(n^{c_d})$, where c_d is exponential in a power of d.

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- $OT_d(n) \leq R^{semi}_{d+1}(n) \leq t_d(n^{c_d})$, where c_d is exponential in a power of d.

Theorem (Suk 2013+)

For $d \geq 2$, we have

$$OT_d(n) \leq t_d(O(n))$$

Lower bound: $OT_3(n) \geq 2^{2^{\Omega(n)}}$ (Elias-Matousek 2012).

Theorem (Suk 2013+)

For $d \geq 2$, we have

$$OT_d(n) \leq t_d(O(n))$$

Corollary

$OT_3(n) = 2^{2^{\Theta(n)}}$

- $OT_1(n) = \Theta(n^2)$ (Erdős-Szekeres 1935)
- $OT_2(n) = 2^{\Theta(n)}$ (Erdős-Szekeres 1935/1960)
- $OT_3(n) = 2^{2^{\Theta(n)}}$ (Elias-Matousek 2012, Suk 2013+)
- For $d \geq 4$, $2^{2^{\Omega(n)}} \leq OT_d(n) \leq t_d(O(n))$ (Elias-Matousek 2012, Suk 2013+)

Lemma

For $d \geq 2$, *we have*

$$OT_d(n) \lesssim 2^{OT_{d-1}(n)}.$$

Since $OT_2(n) = 2^{\Theta(n)}$, $OT_3(n) \leq 2^{2^{O(n)}}$, $OT_4(n) \leq 2^{2^{2^{O(n)}}}$,......

Theorem (Suk 2013+)

For $d \geq 2$, *we have*

$$OT_d(n) \leq t_d(O(n))$$

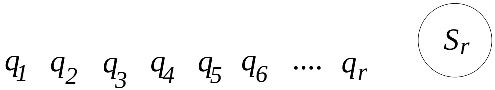
Lemma

For $d \geq 2$ and $M = OT_{d-1}(n)$, we have

$$OT_d(n) < 2^{4d^2 M \log M}.$$

Proof. Set $N = 2^{4d^2 M \log M}$, let $P = p_1, ..., p_N \subset \mathbb{R}^d$. Find subsequence $q_1, q_2, ..., q_r$, and subset $S_r \subset P$, such that for

- For $i < j$, q_i comes before q_j in the original sequence.



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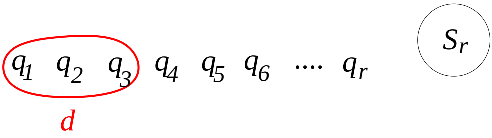

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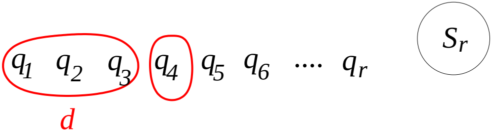

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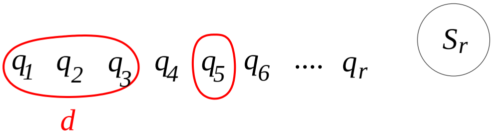

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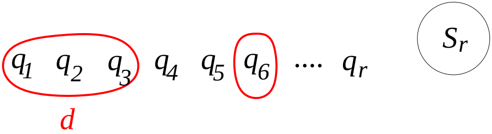

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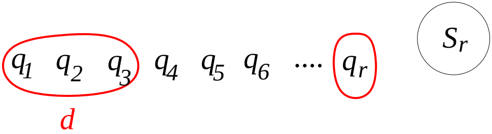

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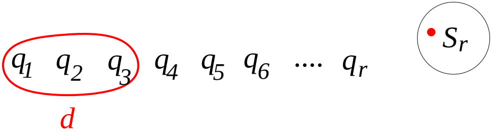

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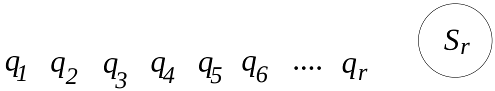
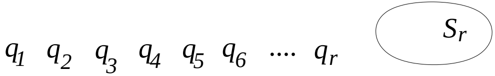


Start: $q_1, q_2, ..., q_{d-1} = p_1, p_2, ..., p_{d-1}$ and $S_{d-1} = P \setminus \{p_1, p_2, ..., p_{d-1}\}$

Inductive Step: Given $q_1, ..., q_r$ and S_r with the 3 properties, we need to find $q_1, ..., q_r, q_{r+1}$ and S_{r+1} with the 3 properties.

q_{r+1} smallest indexed element in S_r. Order is preserved.



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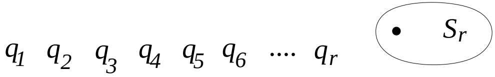

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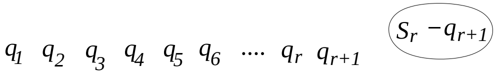

Every d-tuple in $\{q_1, ..., q_{r+1}\}$ gives rise to a hyperplane.

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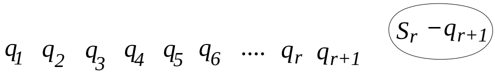

Need the second property, Every d-tuples $(q_{i_1}, ..., q_{i_d})$, $i_1 < i_2 < ... < i_d$, $(q_{i_1}, ..., q_{i_d}, q)$ same orientation for all $q \in \{q_j : i_d < j \leq r+1\} \cup S_{r+1}$.

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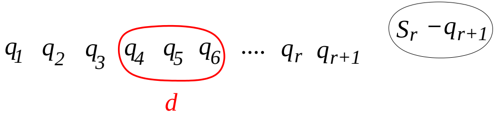

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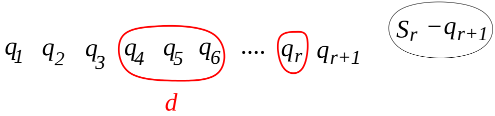

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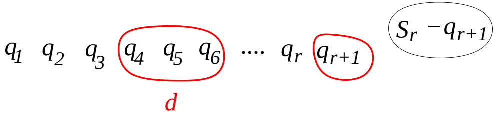

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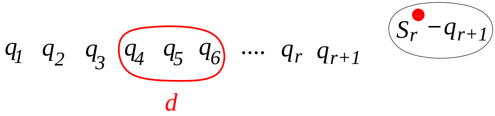

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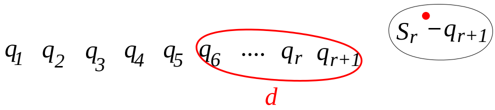

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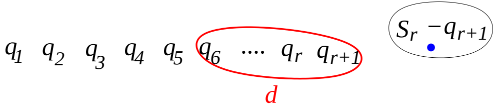

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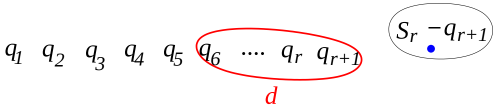
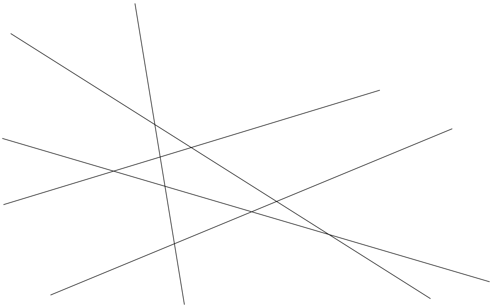


Gives rise to $\binom{r}{d-1}$ hyperplanes in $\mathbb{R}^d$

Hyperplane arrangement lemma: $\binom{r}{d-1}$ hyperplanes divides $\mathbb{R}^d$ into $O(r^{d^2})$ cells (d-faces).



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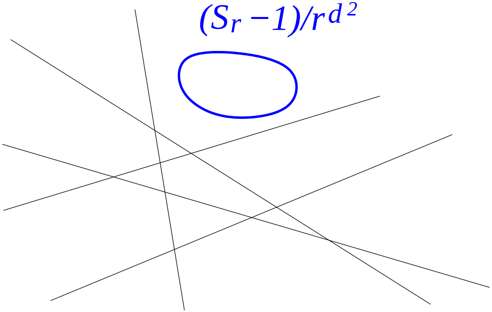

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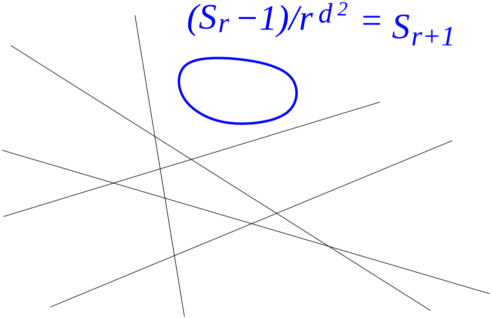
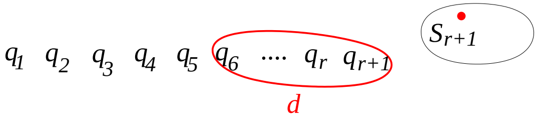


Now, every d-tuples $(q_{i_1}, ..., q_{i_d})$, $i_1 < i_2 < ... < i_d$, $(q_{i_1}, ..., q_{i_d}, q)$ same orientation for all $q \in \{q_j : i_d < j \leq r+1\} \cup S_{r+1}$.

Second condition satisfied.



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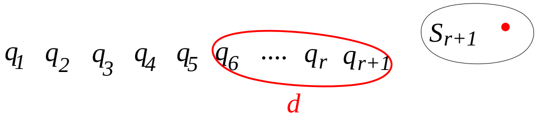

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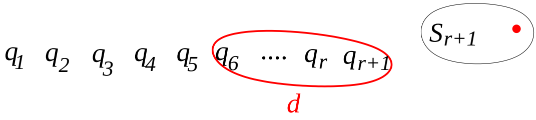

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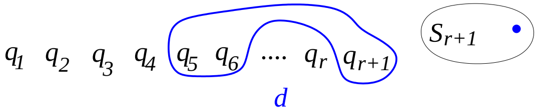

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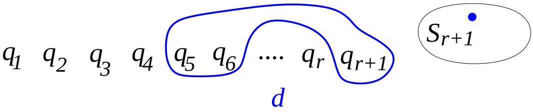

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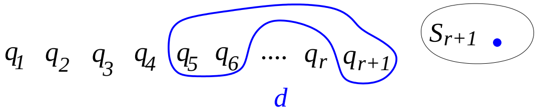

Final condition: $|S_{r+1}| \geq \frac{|S_r|-1}{r^{d^2}}$.

Recall: $|S_r| \geq \frac{N}{((r-1)!)^{d^2}} - r$, implies

$$|S_{r+1}| \geq \frac{N}{(r!)^{d^2}} - (r+1)$$

Hence we can construct the sequence $q_1, q_2, ..., q_{r+1}$ and S_{r+1} with the 3 desired properties.

$N = 2^{4d^2 M \log M}$, for $M = OT_{d-1}(n)$,

$$|S_{2M}| \geq \frac{2^{4d^2 M \log M}}{((2M-2)!)^{d^2}} - 2M > 1$$

$q_1, q_2, ..., q_{2M}$ is well defined.

We have $q_1, q_2, ..., q_{2M}$, where $M = OT_{d-1}(n)$

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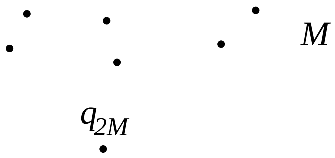

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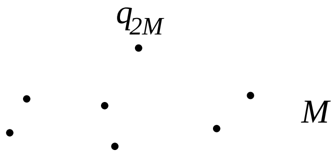

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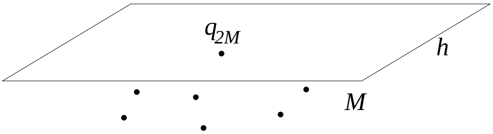

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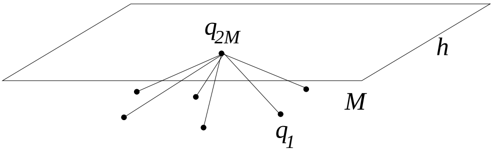

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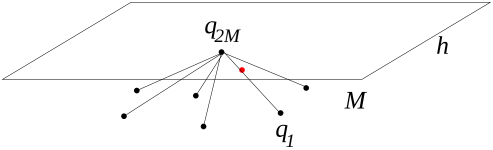
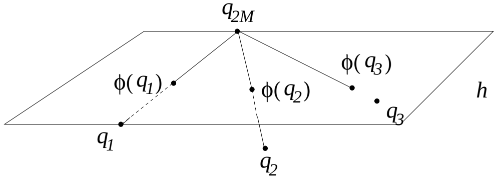


We have $q_1, q_2, ..., q_{2M}$, where $M = OT_{d-1}(n)$

Assume M points from $q_1, .., q_{2M}$ lies below the point q_{2M}



$OT_{d-1}(n)$ points in h. Q' n points every d-tuple in $h = \mathbb{R}^{d-1}$ has the same orientation.

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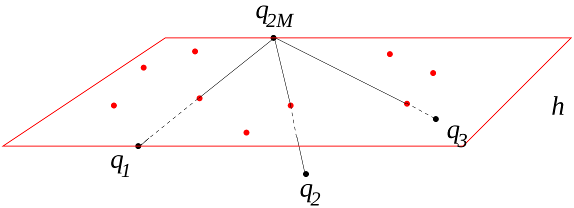

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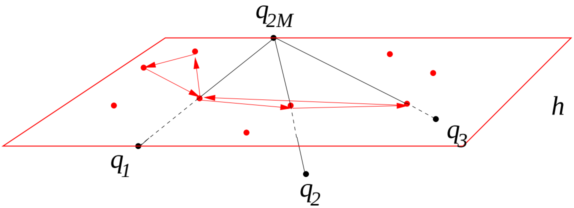

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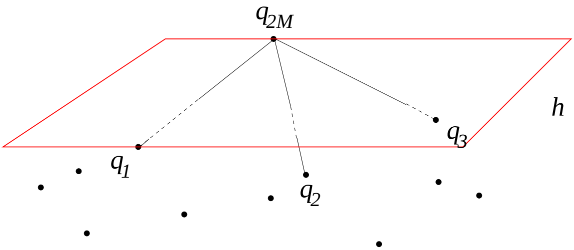

$q_1, q_2, ..., q_n$.

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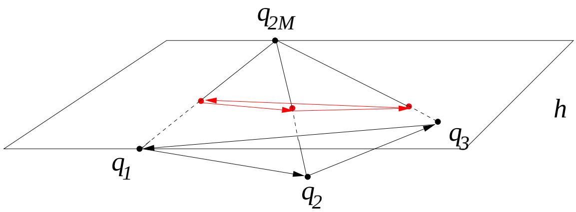
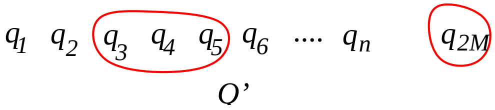
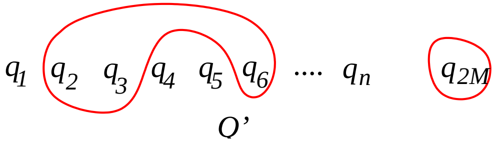


$Q' \cup q_M$: Every $d+1$ tuple with q_{2M} has the same orientation.

$q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5 \quad q_6 \quad \cdots \quad q_n \qquad q_{2M}$

Q'


$q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5 \quad q_6 \quad \cdots \quad q_n \qquad q_{2M}$
Q'


q_1 q_2 q_3 q_4 q_5 q_6 .... q_n q_{2M}
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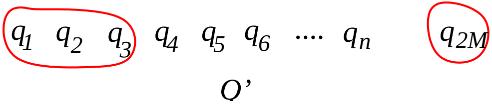
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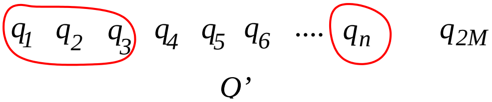
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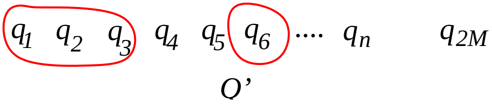
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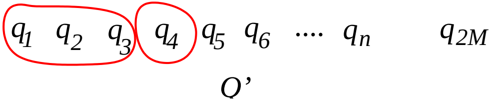
$$Q'$$

$q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5 \quad q_6 \quad \cdots \quad q_n \qquad q_{2M}$

Q'



Every $d+1$-tuple among these points have the same orientation.

Close the gap on $OT_d(n)$ for $d \geq 4$.

- $OT_1(n) = \Theta(n^2)$ (Erdős-Szekeres 1935)
- $OT_2(n) = 2^{\Theta(n)}$ (Erdős-Szekeres 1935/1960)
- $OT_3(n) = 2^{2^{\Theta(n)}}$ (Elias-Matousek 2012, Suk 2013+)
- For $d \geq 4$, $2^{2^{\Omega(n)}} \leq OT_d(n) \leq t_d(O(n))$ (Elias-Matousek 2012, Suk 2013+)

Thank you!