A Ramsey-type result for geometric k-hypergraphs

Dhruv Mubayi and Andrew Suk

September 21, 2013

Old theorems in Ramsey Theory

For k-uniform hypergraphs.

Definition

We define the *Ramsey number* $R_k(n)$ to be the minimum integer N such that any N-vertex k-uniform hypergraph H contains either a clique or an independent set of size n.

Theorem (Ramsey 1930)

For all k, n, the Ramsey number $R_k(n)$ is finite.

Estimate $R_k(n)$, k fixed and $n \to \infty$.

Known estimates

Theorem (Erdős-Szekeres 1935, Erdős 1947)

$$2^{n/2} \leq R_2(n) \leq 2^{2n}.$$

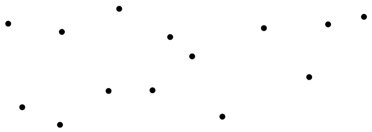
Theorem (Erdős-Rado 1952, Erdős-Hajnal 1960's)

$$2^{cn^2} \leq R_3(n) \leq 2^{2^{c'n}}.$$

$$\mathrm{twr}_{k-1}(cn^2) \leq R_k(n) \leq \mathrm{twr}_k(c'n).$$

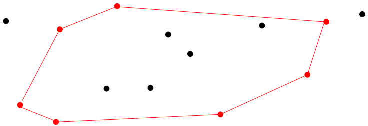
Tower function $\mathrm{twr}_i(x)$ is given by $\mathrm{twr}_1(x) = x$ and $\mathrm{twr}_{i+1}(x) = 2^{\mathrm{twr}_i(x)}$.

Erdős conjecture: $R_3(n) = 2^{2^{\Theta(n)}}$ (offered \$500).



Problem (Esther Klein 1930's)

Given an integer n, does there exist a number ES(n), such that any set of at least ES(n) points in the plane in general position, contains n members in convex position?



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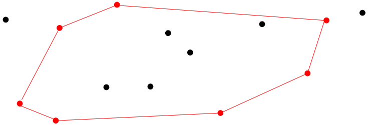
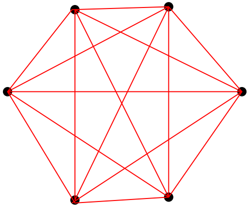


Theorem (Erdős-Szekeres 1935)

$$2^{n-2} + 1 \leq ES(n) \leq \binom{2n-4}{n-2} + 1 = O(4^n/\sqrt{n}).$$

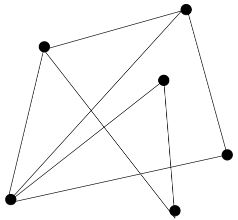
Main Problem (Mubayi and Suk): Combine the **Ramsey problem** on graphs (k-uniform hypergraphs) and the **Erdős-Szekeres problem** on finding points in convex position.

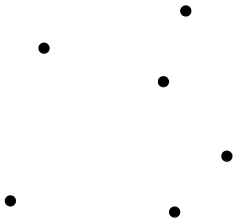
Fits nicely in the **Theory of Geometric Graphs**.



Geometric graphs: Graphs drawn in the plane, whose vertices are represented by points, and edges are represented by straight-line segments connecting the corresponding points.

Geometric k-hypergraphs (in the plane): k-uniform hypergraphs whose vertices are represented by points in the plane, and edges are k-tuples of points.





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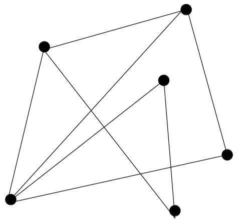

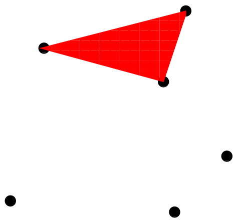

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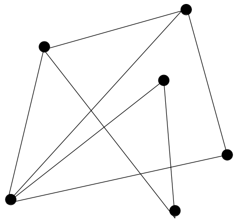

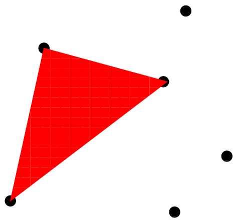

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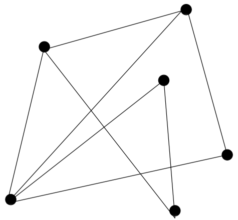

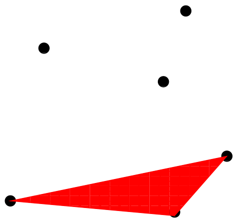

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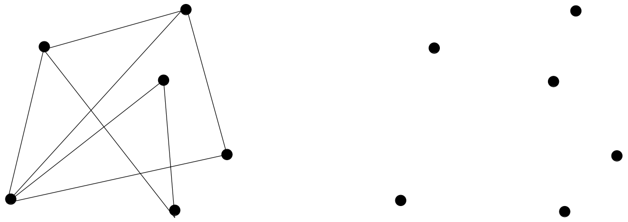
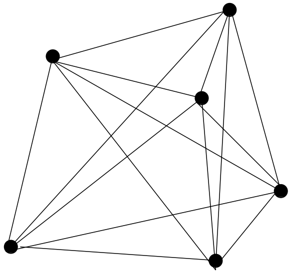


Definition (Mubayi and Suk)

We define the *geometric Ramsey number* $R^{geo}(n) = R_2^{geo}(n)$ to be the minimum integer N such that any N-vertex complete geometric graph whose edges are colored with two colors, must contain a complete monochromatic convex geometric graph on n vertices.



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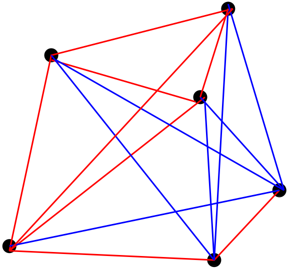

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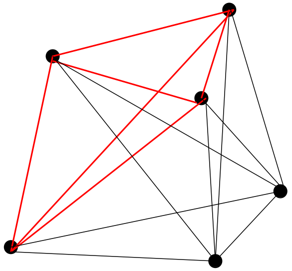

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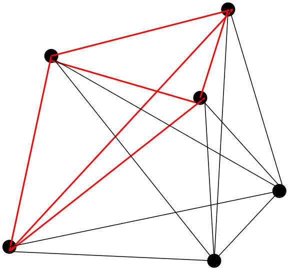

Problem: Estimate $R^{geo}(n)$.

Geometric k-hypergraphs in the plane.

Definition (Mubayi and Suk)

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Trivial bounds on $R_k^{geo}(n)$?

Geometric graphs, $k = 2$
Trivial upper and lower bounds on $R^{geo}(n)$.

Lower bound: $R^{geo}(n) \geq \max\{R(n), ES(n)\} \geq 2^{n-2} + 1$.

Upper bound: $R^{geo}(n) \leq ES(R(n)) \leq 2^{2^{O(n)}}$.

$$2^{n-1} + 1 \leq R^{geo}(n) \leq 2^{2^{O(n)}}.$$

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Similar arguments for geometric k-hypergraphs ($k \geq 3$) shows **double exponential difference**

$$2^{\Omega(n^2)} \leq R_3^{geo}(n) \leq 2^{2^{2^{O(n)}}}$$

$$2^{2^{\Omega(n^2)}} \leq R_4^{geo}(n) \leq 2^{2^{2^{2^{O(n)}}}}$$

$$\mathrm{twr}_{k-1}(\Omega(n^2)) \leq R_k^{geo}(n) \leq \mathrm{twr}_{k+1}(O(n)).$$

Theorem (Mubayi and Suk 2013)

For geometric graphs, we have

$$4^n < R^{geo}(n) < 2^{O(n^2 \log n)}.$$

For geometric 3*-hypergraphs, we have*

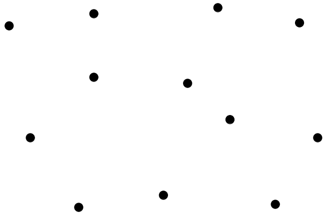
$$R_3^{geo}(n) = 2^{2^{\Theta(n)}}.$$

For geometric k-hypergraphs, $k \geq 4$, we have

$$\mathrm{twr}_{k-1}(\Omega(n^2)) < R_k^{geo}(n) < \mathrm{twr}_k(O(n)).$$

Recall Ramsey numbers for graphs: $R_2(n) = 2^{\Theta(n)}$,
3-hypergraphs: $2^{\Omega(n^2)} < R_3(n) < 2^{2^{O(n)}}$. ($500 problem)
k-hypergraphs: $\mathrm{twr}_{k-1}(\Omega(n^2)) < R_k(n) < \mathrm{twr}_k(O(n))$.

Geometric graphs: $G = (V, E)$ complete geometric graph on with $N = 2^{10n^2 \log n}$ vertices.



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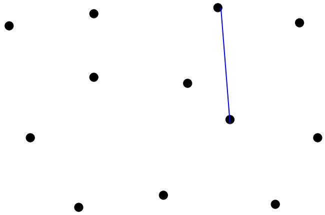

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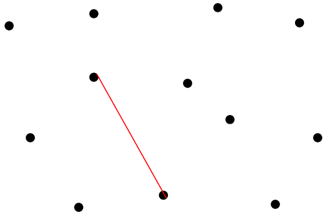

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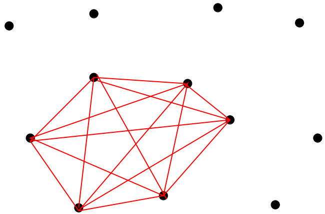
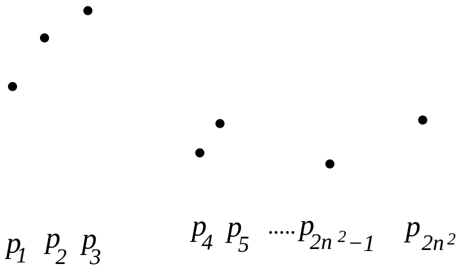
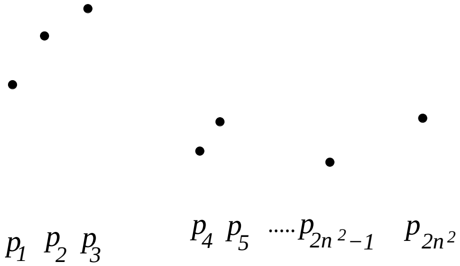


We obtain subset of $2n^2$ vertices



1) Points $p_1, \ldots, p_{2n^2}$ are ordered from left to right.

We obtain subset of $2n^2$ vertices



2) for every pair of vertices p_i and p_j, where $i < j$, all points $p \in \{p_k : k > j\}$ lie above (below) the line $l = p_i p_j$.

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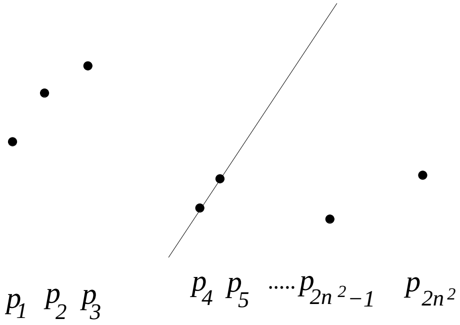

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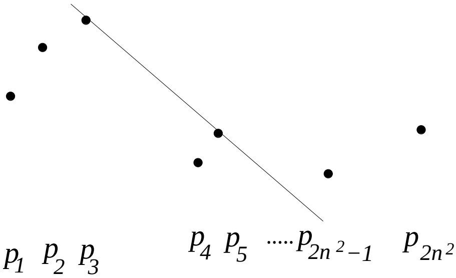

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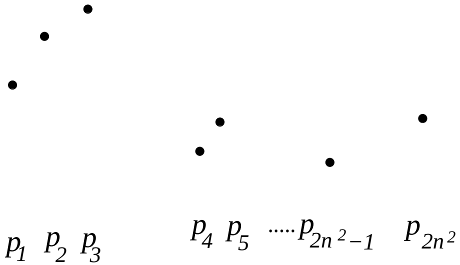

3) for any vertex p_i, all pairs (p_i, p) where $p \in \{p_j : j > i\}$ have the same color.

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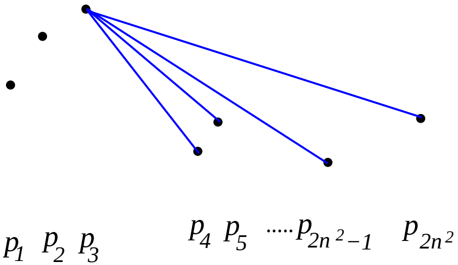

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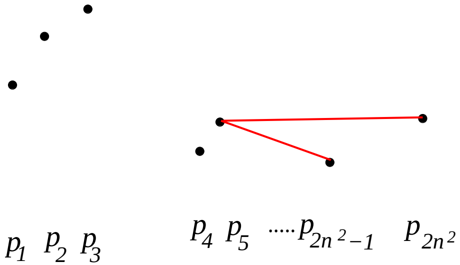

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Greedy algorithm.

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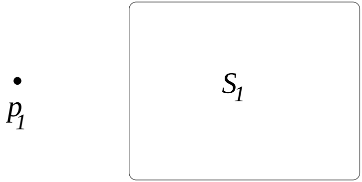
Greedy algorithm. $S_1 = V \setminus \{p_1\}$



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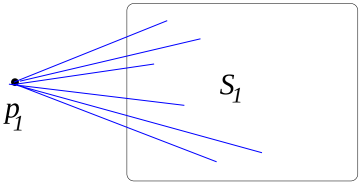

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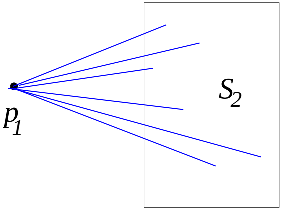

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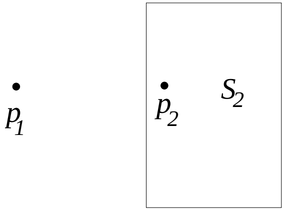

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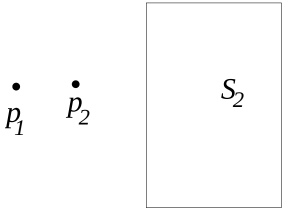

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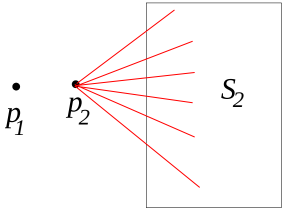

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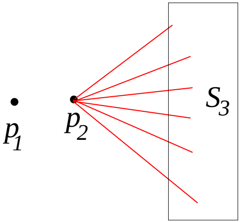

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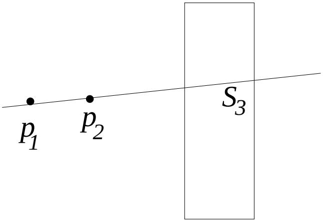

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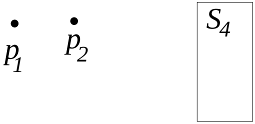

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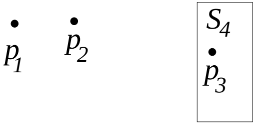

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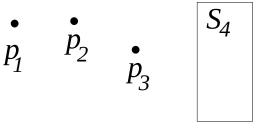
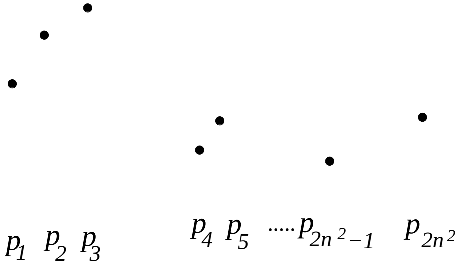
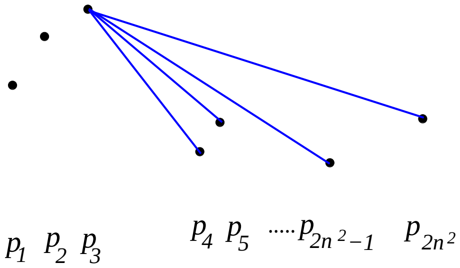
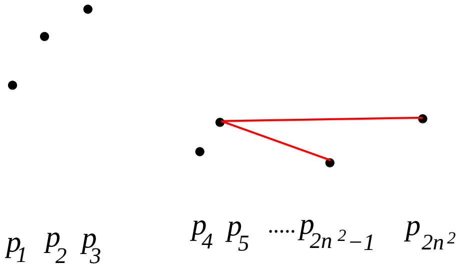


$|V| = 2^{10n^2 \log n}$





p_1 p_2 p_3
p_4 p_5 ..... p_{2n^2-1} p_{2n^2}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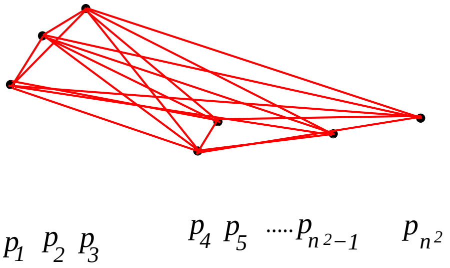
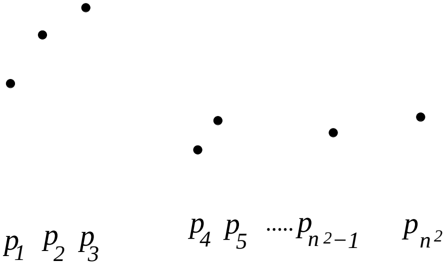
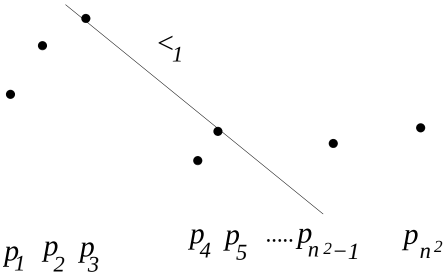


Half of the points form a monochromatic clique (of size n^2). Say $p_1, \ldots, p_{n^2}$.



Define partial orders $\prec_1, \prec_2$:



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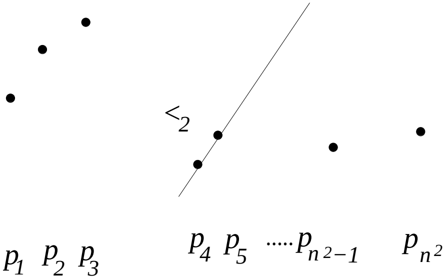

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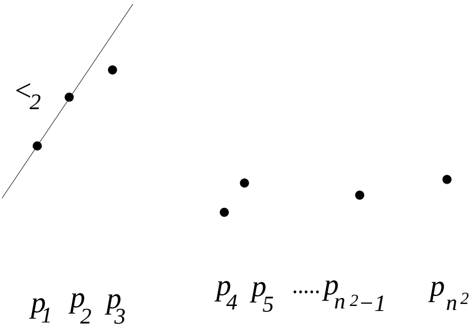

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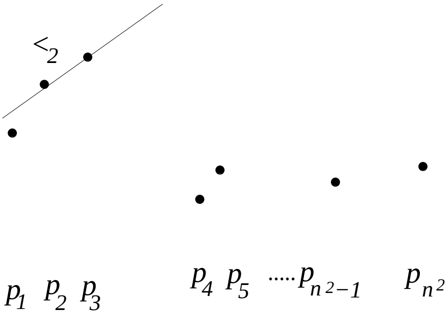

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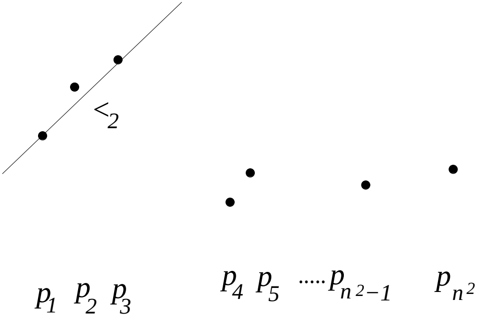
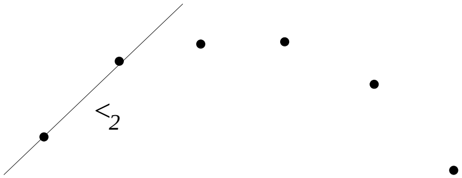


Define partial orders $\prec_1, \prec_2$:

Dilworth's Theorem: Chain of length n with respect to $\prec_1$ or $\prec_2$.



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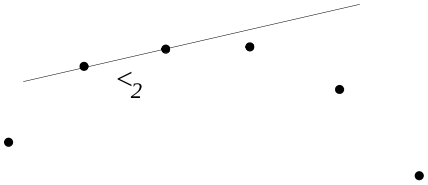

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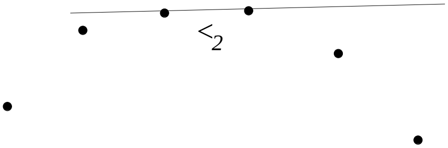

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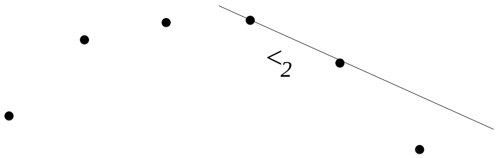

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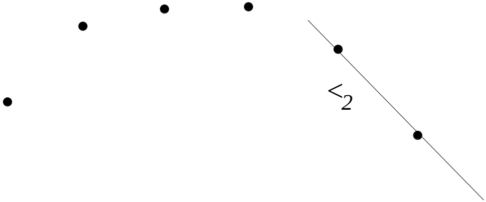

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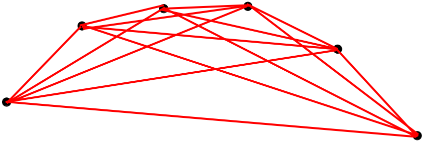

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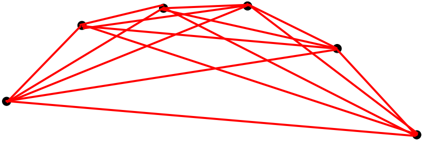

Obtained a monochromatic convex geometric graph on n vertices.

$\square$

Theorem (Mubayi and Suk 2013)

For geometric graphs, we have

$$4^n < R^{geo}(n) < 2^{O(n^2 \log n)}.$$

For geometric 3-hypergraphs, we have

$$R_3^{geo}(n) = 2^{2^{\Theta(n)}}.$$

For geometric k-hypergraphs, $k \geq 4$, we have

$$\mathrm{twr}_{k-1}(\Omega(n^2)) < R_k^{geo}(n) < \mathrm{twr}_k(O(n)).$$

Problems:

1. $R^{geo}(n) = 2^{\Theta(n)}$?
2. Close the gap for $R_k^{geo}(n)$.

Thank you!