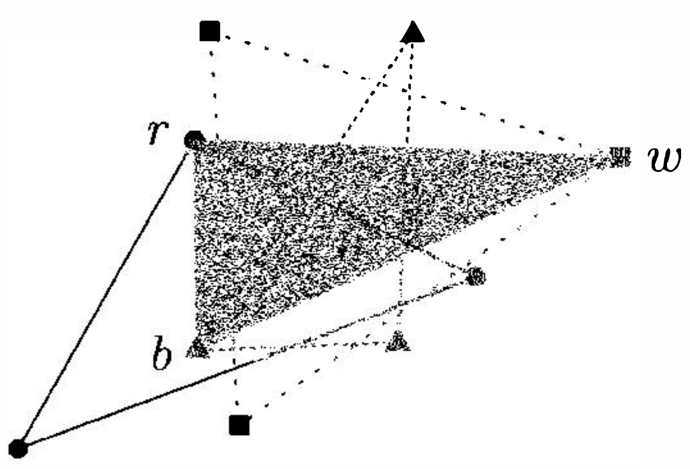
are the complexes where the homology of dimension d and larger vanishes for all induced subcomplexes). We do not formulate it but mention one of its consequences, the *upper bound theorem for families of convex sets*: If $f_r(\mathcal{N}(\mathcal{F})) = 0$ for a family $\mathcal{F}$ of n convex sets in $\mathbf{R}^d$ and some r, $d \leq r \leq n$, then $f_k(\mathcal{N}(\mathcal{F})) \leq \sum_{j=0}^{d} \binom{r-d}{k-j+1}\binom{n-r+d}{j}$; equality holds, e.g., in the case mentioned above (several copies of $\mathbf{R}^d$ and hyperplanes in general position).

Exercises

1. Show that any compact set in $\mathbf{R}^d$ has a unique point with the lexicographically smallest coordinate vector. $\boxed{3}$
2. Prove the following *colored Helly theorem*: *Let $\mathcal{C}_1, \ldots, \mathcal{C}_{d+1}$ be finite families of convex sets in $\mathbf{R}^d$ such that for any choice of sets $C_1 \in \mathcal{C}_1, \ldots, C_{d+1} \in \mathcal{C}_{d+1}$, the intersection $C_1 \cap \cdots \cap C_{d+1}$ is nonempty. Then for some i, all the sets of $\mathcal{C}_i$ have a nonempty intersection.* Apply a method similar to the proof of the fractional Helly theorem; i.e., consider the lexicographic minima of the intersections of suitable collections of the sets. $\boxed{5}$
   The result is due to Lovász ([Lov74]; also see [Bár82]).
3. Let $F_1, F_2, \ldots, F_n$ be convex sets in $\mathbf{R}^d$. Prove that there exist convex polytopes $P_1, P_2, \ldots, P_n$ such that $\dim(\bigcap_{i\in I} F_i) = \dim(\bigcap_{i\in I} P_i)$ for every $I \subseteq \{1, 2, \ldots, n\}$ (where $\dim(\emptyset) = -1$). $\boxed{2}$

8.2 The Colorful Carathéodory Theorem

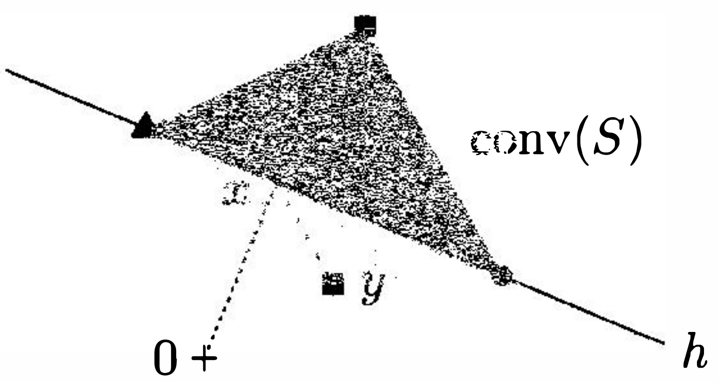
Carathéodory's theorem asserts that if a point x is in the convex hull of a set $X \subseteq \mathbf{R}^d$, then it is in the convex hull of some at most $d+1$ points of X. Here we present a "colored version" of this statement. In the plane, it shows the following: Given a red triangle, a blue triangle, and a white triangle, each of them containing the origin, there is a vertex r of the red triangle, a vertex b of the blue triangle, and a vertex w of the white triangle such that the tricolored triangle rbw also contains the origin. (In the following pictures, the colors of points are distinguished by different shapes of the point markers.)



The d-dimensional statement follows.

8.2.1 Theorem (Colorful Carathéodory theorem). *Consider $d+1$ finite point sets $M_1, \ldots, M_{d+1}$ in $\mathbf{R}^d$ such that the convex hull of each M_i contains the point 0 (the origin). Then there exists a $(d+1)$-point set $S \subseteq M_1 \cup \cdots \cup M_{d+1}$ with $|M_i \cap S| = 1$ for each i and such that $0 \in \operatorname{conv}(S)$. (If we imagine that the points of M_i have "color" i, then we look for a "rainbow" $(d+1)$-point S with $0 \in \operatorname{conv}(S)$, where "rainbow" = "containing all colors.")*

Proof. Call the convex hull of a $(d+1)$-point rainbow set a *rainbow simplex*. We proceed by contradiction: We suppose that no rainbow simplex contains 0, and we choose a $(d+1)$-point rainbow set S such that the distance of $\operatorname{conv}(S)$ to 0 is the smallest possible. Let x be the point of $\operatorname{conv}(S)$ closest to 0. Consider the hyperplane h containing x and perpendicular to the segment $0x$, as in the picture:



Then all of S lies in the closed half-space h^- bounded by h and not containing 0. We have $\operatorname{conv}(S) \cap h = \operatorname{conv}(S \cap h)$, and by Carathéodory's theorem, there exists an at most d-point subset $T \subseteq S \cap h$ such that $x \in \operatorname{conv}(T)$.

Let i be a color not occurring in T (i.e., $M_i \cap T = \emptyset$). If all the points of M_i lay in the half-space h^-, then 0 would not be in $\operatorname{conv}(M_i)$, which we assume. Thus, there exists a point $y \in M_i$ lying in the complement of h^- (strictly, i.e., $y \notin h$).

Let us form a new rainbow set S' from S by replacing the (unique) point of $M_i \cap S$ by y. We have $T \subset S'$, and so $x \in \operatorname{conv}(S')$. Hence the segment xy is contained in $\operatorname{conv}(S')$, and we see that $\operatorname{conv}(S')$ lies closer to 0 than $\operatorname{conv}(S)$, a contradiction. The colorful Carathéodory theorem is proved. □

This proof suggests an algorithm for finding the rainbow simplex as in the theorem. Namely, start with an arbitrary rainbow simplex, and if it does not contain 0, switch one vertex as in the proof. It is not known whether the number of steps of this algorithm can be bounded by a polynomial function of the dimension and of the total number of points in the M_i. It would be very interesting to construct configurations where the number of steps is very large or to prove that it cannot be too large.

Bibliography and remarks. The colorful Carathéodory theorem is due to Bárány [Bár82]. Its algorithmic aspects were investigated by Bárány and Onn [BO97].

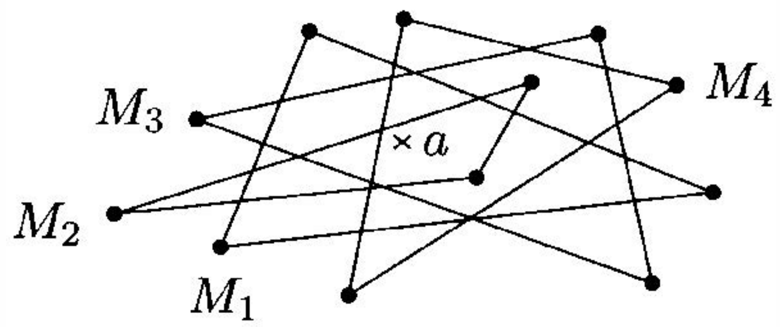
For easier formulations we introduce the following terminology: If $X \subset \mathbf{R}^d$ is a finite set, an *X-simplex* is the convex hull of some $(d+1)$-tuple of points of X. We make the convention that X-simplices are in bijective correspondence with their vertex sets. This means that two X-simplices determined by two distinct $(d+1)$-point subsets of X are considered different even if they coincide as subsets of $\mathbf{R}^d$. Thus, the X-simplices form a multiset in general. This concerns only sets X in degenerate positions; if X is in general position, then distinct $(d+1)$-point sets have distinct convex hulls.

9.1.1 Theorem (First selection lemma). *Let X be an n-point set in $\mathbf{R}^d$. Then there exists a point $a \in \mathbf{R}^d$ (not necessarily belonging to X) contained in at least $c_d\binom{n}{d+1}$ X-simplices, where $c_d > 0$ is a constant depending only on the dimension d.*

The best possible value of c_d is not known, except for the planar case. The first proof below shows that for n very large, we may take $c_d \approx (d+1)^{-(d+1)}$.

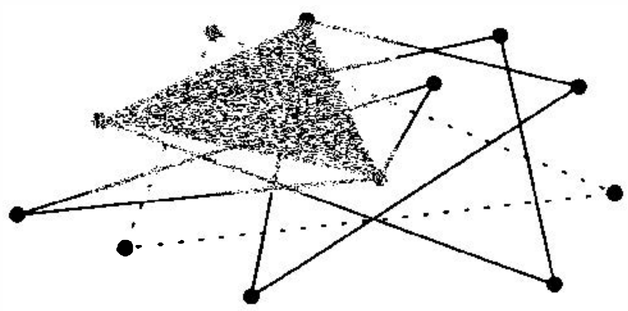
The first proof: from Tverberg and colorful Carathéodory. We may suppose that n is sufficiently large ($n \geq n_0$ for a given constant n_0), for otherwise, we can set c_d to be sufficiently small and choose a point contained in a single X-simplex.

Put $r = \lceil n/(d+1) \rceil$. By Tverberg's theorem (Theorem 8.3.1), there exist r pairwise disjoint sets $M_1, \ldots, M_r \subseteq X$ whose convex hulls all have a point in common; call this point a. (A typical M_i has $d+1$ points, but some of them may be smaller.)



We want show that the point a is contained in many X-simplices (so far we have $const \cdot n$ and we need $const \cdot n^{d+1}$).

Let $J = \{j_0, \ldots, j_d\} \subseteq \{1, 2, \ldots, r\}$ be a set of $d+1$ indices. We apply the colorful Carathéodory's theorem (Theorem 8.2.1) for the $(d+1)$ "color" sets $M_{j_0}, \ldots, M_{j_d}$, which all contain a in their convex hull. This yields a rainbow X-simplex S_J containing a and having one vertex from each of the M_{j_i}, as illustrated below:



If $J' \neq J$ are two $(d+1)$-tuples of indices, then $S_J \neq S_{J'}$. Hence the number of X-simplices containing the point a is at least

$$\binom{r}{d+1} = \binom{\lceil n/(d+1) \rceil}{d+1} \geq \frac{1}{(d+1)^{d+1}} \frac{n(n-(d+1))\cdots(n-d(d+1))}{(d+1)!}.$$

For n sufficiently large, say $n \geq 2d(d+1)$, this is at least $(d+1)^{-(d+1)}2^{-d}\binom{n}{d+1}$. □

The second proof: from fractional Helly. Let $\mathcal{F}$ denote the family of all X-simplices. Put $N = |\mathcal{F}| = \binom{n}{d+1}$. We want to apply the fractional Helly theorem (Theorem 8.1.1) to $\mathcal{F}$. Call a $(d+1)$-tuple of sets of $\mathcal{F}$ *good* if its $d+1$ sets have a common point. To prove the first selection lemma, it suffices to show that there are at least $\alpha\binom{N}{d+1}$ good $(d+1)$-tuples for some $\alpha > 0$ independent of n, since then the fractional Helly theorem provides a point common to at least βN members of $\mathcal{F}$.

Set $t = (d+1)^2$ and consider a t-point set $Y \subset X$. Using Tverberg's theorem, we find that Y can be partitioned into $d+1$ pairwise disjoint sets, of size $d+1$ each, whose convex hulls have a common point. (Tverberg's theorem does not guarantee that the parts have size $d+1$, but if they don't, we can move points from the larger parts to the smaller ones, using Carathéodory's theorem.) Therefore, each t-point $Y \subset X$ provides at least one good $(d+1)$-tuple of members of $\mathcal{F}$. Moreover, the members of this good $(d+1)$-tuple are pairwise vertex-disjoint, and therefore the $(d+1)$-tuple uniquely determines Y. It follows that the number of good $(d+1)$-tuples is at least $\binom{n}{t} = \Omega(n^{(d+1)^2}) \geq \alpha\binom{N}{d+1}$. □

In the first proof we have used Tverberg's theorem for a large point set, while in the second proof we applied it only to configurations of bounded size. For the latter application, if we do not care about the constant of proportionality in the first selection lemma, a weaker version of Tverberg's theorem suffices, namely the finiteness of $T(d, d+1)$, which can be proved by quite simple arguments, as we have seen.

The relation of Tverberg's theorem to the first selection lemma in the second proof somewhat resembles the derivation of macroscopic properties in physics (pressure, temperature, etc.) from microscopic properties (laws of motion of molecules, say). From the information about small (microscopic) configurations we obtained a global (macroscopic) result, saying that a significant portion of the X-simplices have a common point.

A point in the interior of many X-simplices. In applications of the first selection lemma (or its relatives) we often need to know that there is a point contained in the *interior* of many of the X-simplices. To assert anything like that, we have to assume some kind of nondegenerate position of X. The following lemma helps in most cases.

9.1.2 Lemma. *Let $X \subset \mathbf{R}^d$ be a set of $n \geq d+1$ points in general position, meaning that no $d+1$ points of X lie on a common hyperplane, and let $\mathcal{H}$ be the set of the $\binom{n}{d}$ hyperplanes determined by the points of X. Then no point*