Practice Test 2

MCS 421 Combinatorics

Problem 1. Determine the number of permutations of $\{1, 2, \ldots, 8\}$ in which no even integer is in its natural position.

Problem 2. Prove that D_n, the number of derangements of $\{1, 2, \ldots, n\}$, is an even number if and only if n is an odd number. Hint: Recall $D_n = nD_{n-1} + (-1)^n$.

Problem 3. Use combinatorial reasoning to derive the identity

$$n! = \binom{n}{0} D_n + \binom{n}{1} D_{n-1} + \cdots + \binom{n}{n+1} D_1 + \binom{n}{n} D_0.$$

Problem 4. Prove that the nth Fibonacci number f_n is the integer that is closest to the number

$$\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n.$$

Problem 5. Let $h_0, h_1, \ldots, h_n, \ldots$ be the sequence defined by $h_n = n^3$ for $n \geq 0$. Show that

$$h_n = h_{n-1} + 3n^2 - 3n + 1,$$

is the recurrence relation for the sequence.

Problem 6. Determine the number of ways to color the squares of a 1-by-n chessboard, using colors red, blue, green, and orange if an even number of squares is to be colored red and an even number is to be colored green.

Problem 7. Solve the recurrence relation $h_n = 4h_{n-2}$ $(n \geq 2)$ with initial values $h_0 = 0$ and $h_1 = 1$.

Problem 8. Solve the nonhomogeneous recurrence relations $h_n = 3h_{n-1} - 2$ for $n \geq 1$ and $h_0 = 1$.

Problem 9. Prove that the number of paths starting at $(0, 0)$ and ending at (n, n) on the $n \times n$ grid, where at each step you movie 1 unit up or 1 unit to the right, and always stay on or below the $y = x$ line, is exactly $\frac{1}{n+1}\binom{2n}{n}$.