Practice Final 2

MCS 421 Combinatorics

Problem 1. Prove that the Stirling numbers of the second kind satisfy a) $S(n, 2) = 2^{n-1} - 1$ for $n \geq 2$, b) $S(n, n-1) = \binom{n}{2}$ for $n \geq 1$. Hint: Use the recurrence $S(p, k) = kS(p-1, k) + S(p-1, k-1)$.

Problem 2. Prove that Stirling numbers of the first kind satisfy a) $s(n, 1) = (n-1)!$ for $n \geq 1$ and b) $s(n, n-1) = \binom{n}{2}$ for $n \geq 1$. Hint: Use the recurrence $s(p, k) = (p-1)s(p-1, k) + s(p-1, k-1)$.

Problem 3. Let $s^{\#}(p, k)$ denote the number of ways to arrange p people in k nonempty circles. Prove that $s^{\#}(p, p) = 1$ and $s^{\#}(p, 0) = 0$ for $p \geq 1$, and

$$s^{\#}(p, k) = (p-1)s^{\#}(p-1, k) + s^{\#}(p-1, k-1).$$

Problem 4. Compute $1^3 + 2^3 + \cdots + n^3 = \sum_{k=0}^{n} k^4$, that is, put it in closed form.

Problem 5. A collection of subsets of $\{1, 2, ..., n\}$ has the property that each pair of subsets has at least one element in common. Prove that there are at most 2^{n-1} subsets in the collection.

Problem 6. Evaluate the sum $\sum_{k=0}^{n} \binom{n}{k} 2^k$.