On order types of systems of segments in the plane

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Abstract

Let $r(n)$ denote the largest integer such that every family $\mathcal{C}$ of n pairwise disjoint segments in the plane in general position has $r(n)$ members whose order type can be represented by points. Pach and Tóth gave a construction that shows $r(n) < n^{\log 8/\log 9}$ [11]. They also stated that one can apply the Erdős-Szekeres theorem for convex sets in [10] to obtain $r(n) > \log_{16} n$. In this note, we will show that $r(n) > cn^{1/4}$ for some absolute constant c.

Introduction

We say that n pairwise disjoint convex sets $\mathcal{C}$ are in *general position* if no three have a common tangent and for every distinct members $A, B, C \in \mathcal{C}$, $conv(A \cup B \cup C) \neq conv(A \cup B)$, that is, C is not a subset of $conv(A \cup B)$. We say that the ordered triple $(A, B, C) \subset \mathcal{C}$ has a *clockwise* (*counterclockwise*) *orientation* if there are three points $a \in A$, $b \in B$, $c \in C$ on the boundary of $conv(A \cup B \cup C)$ that follow each other in clockwise (counterclockwise) order. Note that a triple (A, B, C) may have both orientations. See Figure 1. Finally we say that $\mathcal{C}$ is *representable* by a point set P if there is a bijection $f : \mathcal{C} \to P$ such that if (A, B, C) has a unique orientation then $(f(A), f(B), f(C))$ has the same orientation.

Given a sequence of convex sets $\mathcal{C}$ in the plane in general position, the *order type* of $\mathcal{C}$ is the mapping assigning each triple $(A, B, C) \subset \mathcal{C}$ the orientation of that triple. The order type of a *point set* was introduced by Goodman and Pollack [6] in the early eighties, and has played a significant role in geometric transversal theory [13]. According to the conjecture of Erdős and Szekeres [7], every set of $2^{n-2}+1$ points in general position contains n points in convex position. Bisztriczky and Fejes Tóth [2] generalized this conjecture as follows. Every family of $2^{n-2}+1$ disjoint convex sets in general position has n members in convex position. A. Hubard and L. Montejano suggested a stronger conjecture, that every family of convex sets in general position can be represented by points. However, Pach and Tóth [11] gave a construction of n pairwise disjoint segments in general position with no subfamily of size $n^{\log 8/\log 9}$ whose order type is representable by points. They observed that it follows from a generalization of the Erdős-Szekeres theorem for convex sets [10] that one can find $\log_{16} n$ members whose order type is representable by points. Our main result is:

Theorem 1. *Let $r(n)$ denote the largest integer such that every family $\mathcal{C}$ of n pairwise disjoint segments in the plane in general position has $r(n)$ members whose order type can be represented by points. Then there exists an absolute constant c_1 such that $c_1 n^{1/4} < r(n) < n^{\log 8/\log 9}$.*

The proof of Theorem 1 is based on the following result for line transversals. Recall that a collection of convex sets in the plane $\mathcal{C}$ has a *line transversal* if there is a line that meets all members in $\mathcal{C}$.

Theorem 2. *For any α such that $0 < \alpha < 1$, every family of n convex sets $\mathcal{C}$ in the plane with no three having a common tangent line, has a subfamily $\mathcal{S} \subset \mathcal{C}$ such that either*

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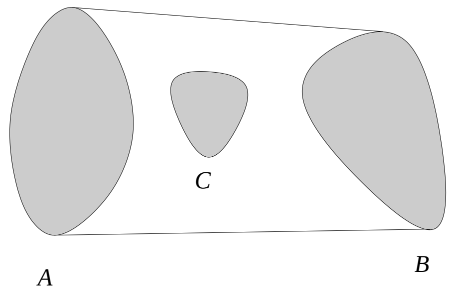
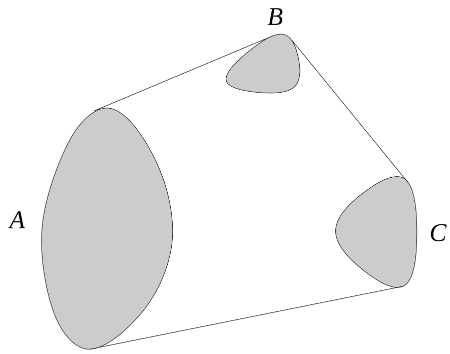
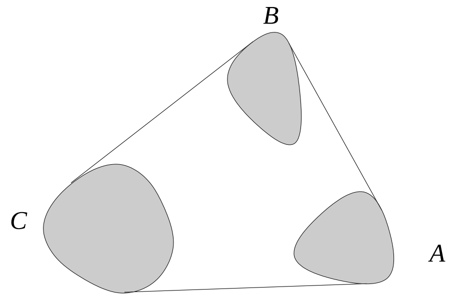


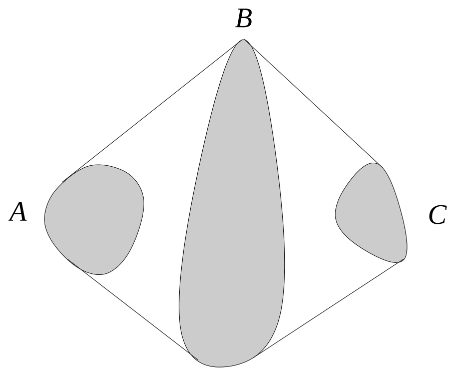
(a) $conv(A, B, C) = conv(A, B)$.



(b) (A, B, C) with a clockwise orientation.



(c) (A, B, C) with a counterclockwise orientation.



(d) (A, B, C) with both a clockwise and a counterclockwise orientation.

Figure 1.

1. *none of the triples in* $\mathcal{S}$ *have a line transversal and* $|\mathcal{S}| \geq \min(c_2\alpha^{-1/2}, (2/3)n)$

2. *or* $\mathcal{S}$ *has a line transversal and* $|\mathcal{S}| \geq c_3\alpha n$*,*

for some absolute constant c_2, c_3*.*

By setting $\alpha = n^{-2/3}$, we have the following corollary

Corollary 3. *Every family of* n *convex sets* $\mathcal{C}$ *in the plane with no three having a common tangent line, has a subfamily* $\mathcal{S} \subset \mathcal{C}$ *with* $|\mathcal{S}| \geq c_4 n^{1/3}$ *such that either*

1. *none of the triples in* $\mathcal{S}$ *have a line transversal*

2. *or* $\mathcal{S}$ *has a line transversal*

for some absolute constant c_4*.*

Proof of Theorem 2

In this section we will prove Theorem 2, which relies on two lemmas.

Lemma 4. *(Spencer [12]) Let* $H = (V, E)$ *be an* r*-uniform hypergraph on* n *vertices. If* $|E(H)| > n/r$*, then there exists a subset* $S \subset V(H)$ *such that* S *is an independent set and*

$$|S| \geq \left(1 - \frac{1}{r}\right) n \left(\frac{n}{r|E(H)|}\right)^{\frac{1}{r-1}}.$$

□

The second lemma is known as the *fractional Helly theorem for line transversals* in [9], and is due to Alon and Kalai [1]. Recall that Helly's theorem states that given a family $\mathcal{C}$ of convex sets in $\mathbb{R}^d$ such that every $d+1$ share a point, then all of $\mathcal{C}$ shares a point. Ever since Helly proved this beautiful theorem back in 1923 [7], there have been a vast number of Helly type results [4]. The first version of the fractional Helly theorem was proved by Katchalski and Liu [8]. We need the following.

Lemma 5. *(Alon and Kalai [1]) Let $\mathcal{C}$ be n convex sets in the plane such that no three share a common tangent. If there are at least $\alpha\binom{n}{3}$ triples with a line transversal, then there exists line that intersects $\frac{\alpha}{25}n$ members in $\mathcal{C}$.*

□

Proof of Theorem 2. Let H be a 3-uniform hypergraph with $V(H) = \mathcal{C} = \{C_1, C_2, ..., C_n\}$ and $\{C_i, C_j, C_k\} \in E(H)$ if and only if there is a line that intersects C_i, C_j, C_k. Notice that an independent set in H corresponds to a subfamily of convex sets with no three having a line transversal. We can assume that $\alpha\binom{n}{3} > n/3$, since otherwise for large enough n we can find a line that intersects at least $c_3\alpha n < 1$ members of $\mathcal{C}$. Now the proof falls into three cases.

Case 1: If $|E(H)| \leq n/3$, then we can get rid of all of the edges by deleting at most $n/3$ vertices. Hence we can find an independent set of size $2n/3$.

Case 2: If $n/3 < |E(H)| \leq \alpha\binom{n}{3}$, then by applying Lemma 4 above, there exists an independent set $S \subset V(H)$ such that

$$|S| \geq \frac{2}{3}n\left(\frac{n}{3\alpha\binom{n}{3}}\right)^{1/2} \geq c_2\alpha^{-1/2}$$

for some absolute constant c_2.

Case 3: If $|E(H)| > \alpha\binom{n}{3}$, then by Lemma 5 we can find a line that intersects at least $c_3\alpha n$ convex sets for some constant c_3.

□

Proof of Theorem 1

As mentioned before, the upper bound comes from a construction by Pach and Tóth [11]. For the lower bound, let $\mathcal{C} = \{S_1, S_2, ..., S_n\}$ be a collection of n segments in the plane. By setting $\alpha = n^{-1/2}$, Theorem 2 implies that there are at least $c_2n^{1/4}$ segments such that no triple has a line transversal or $c_3n^{1/2}$ segments that can all be intersected by some line.

Case 1: If there are at least $c_2n^{1/4}$ segments $\mathcal{S} \subset \mathcal{C}$ such that every triple does not have a line transversal, then the segments "behave" like points. Hence by picking one point from each segment in $\mathcal{S}$, we have a point set that represents the order type of $\mathcal{S}$.

Case 2: Suppose there exist at least $c_3n^{1/2}$ segments $\mathcal{S} \subset \mathcal{C}$ all on a line. Without loss of generality we can assume that this line is the y-axis and no segment is vertical. We order the segments of $\mathcal{S}$ in the order they intersect the y-axis from bottom to top. By the Erdős-Szekeres theorem [7], there exists a subfamily

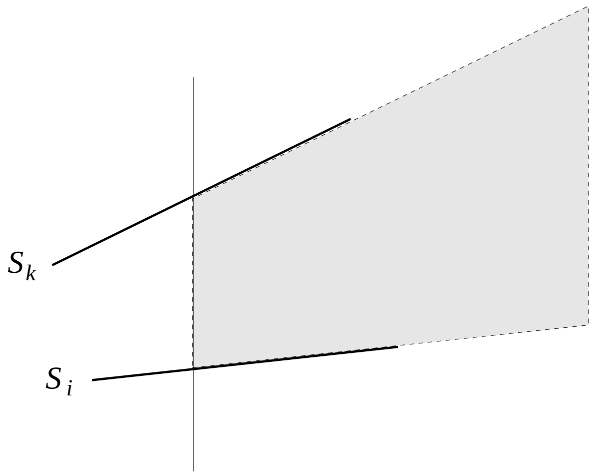


Figure 2: The region where the right endpoint of S_j must lie.

$\mathcal{S}' \subset \mathcal{S}$ with $|\mathcal{S}'| \geq \sqrt{c_2}n^{1/4}$ such that the slopes of the segments are increasing or decreasing from bottom to top. With a slight abuse of notation, let us assume $\mathcal{S}' = \{S_1, S_2, ..., S_{|\mathcal{S}'|}\}$ is ordered from bottom to top and let l_i, r_i denote the left and right endpoints of S_i for each i. If the slopes are increasing in $\mathcal{S}'$, then for any S_i, S_j, S_k with $i < j < k$, r_j must lie in the right half-plane below the line that contains S_k and above the line that contains S_i. See Figure 2.

If (r_i, r_j, r_k) has a counterclockwise orientation, then r_i, r_j, r_k must lie on the boundary of $conv(S_i \cup S_j \cup S_k)$. Therefore (S_i, S_j, S_k) has a counterclockwise orientation (or both). Since $S_j \not\subset conv(S_i \cup S_k)$, if (r_i, r_j, r_k) has a clockwise orientation, then (r_i, l_j, r_k) must lie on the boundary of $conv(S_i \cup S_j \cup S_k)$. Hence (S_i, S_j, S_k) has a clockwise orientation. Therefore the point set $P' = \{r_1, ..., r_{|\mathcal{S}'|}\}$ represents the order type of $\mathcal{S}'$. If the slopes in $\mathcal{S}'$ were decreasing from bottom to top, then by a similar argument, the point set $P' = \{l_1, l_2, ..., l_{|\mathcal{S}'|}\}$ would represent the order type of $\mathcal{S}'$.

□

Conclusion

We would like to make two final remarks. By combining Lemma 4 and Proposition 4.1 in [1], one can easily generalize Corollary 3 for higher dimensions.

Theorem 6. *Every family of n convex sets in $\mathbb{R}^d$ with no $d+1$ have a common tangent has a subfamily $\mathcal{S} \subset \mathcal{C}$ with $|\mathcal{S}| \geq c_d n^{\frac{1}{d+1}}$ such that either*

1. *none of the $(d+1)$-tuples in $\mathcal{S}$ have a hyperplane that meets all of them,*

2. *or there exists a hyperplane that intersects all of S,*

where c_d is a constant that depends only on d.

Since the proof of Theorem 1 relies heavily on Theorem 2, we conjecture the following.

Conjecture 7. *There exists an absolute constant ϵ such that every family of n convex sets in the plane in general position has a subfamily of size n^ϵ whose order type can be represented by points.*

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