Tangencies between disjoint regions in the plane

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June 16, 2010

Problem

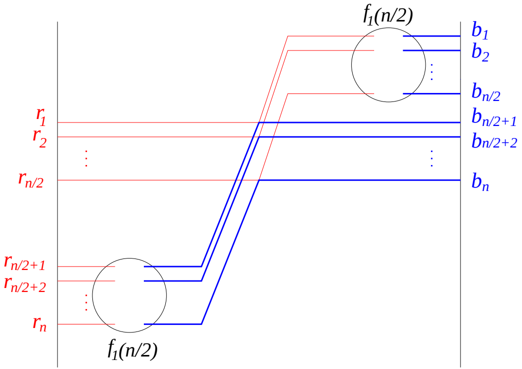
Definition

Two nonoverlapping Jordan regions in the plane are said to *touch* each other or to be *tangent* to each other if their boundaries have precisely one point in common and their interiors are disjoint.

Problem

Given two families $\mathcal{R}, \mathcal{B}$ of closed Jordan regions, each consisting of n pairwise disjoint members, what is the maximum number tangencies between $\mathcal{R}$ and $\mathcal{B}$?

Can be superlinear. Below are two disjoint families of x-monotone curves with at least $\Omega(n \log n)$ tangencies.

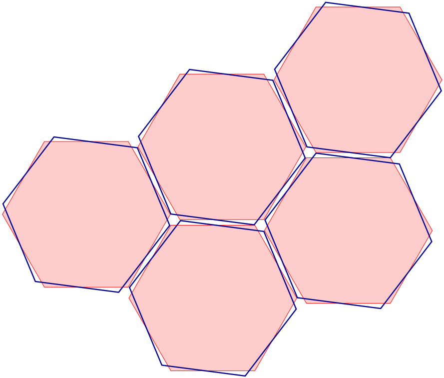


Main Result

Theorem (Pach, Suk, Treml)

The number of tangencies between two families of convex bodies in the plane, each consisting of $n > 2$ pairwise disjoint members, cannot exceed $8n - 16$.

Not far from being optimal. Below has $6n(1 - o(1))$ tangencies by taking n translates of each hexagon and arranging them in a lattice-like fashion.



Corollary (Pach, Suk, Treml)

Let $\mathcal{C}$ be a family of n convex bodies in the plane which can be decomposed into k subfamilies consisting of pairwise disjoint bodies. Then that total number of tangencies between members in $\mathcal{C}$ is $O(kn)$. This bound is tight up to a multiplicative constant.

Conjecture (Pach, Suk, Treml)

For every fixed integer $k > 2$, the number of tangencies in any n-member family of convex bodies, no k of which are pairwise intersecting, is at most $O(kn)$.

Related conjectures

Conjecture (Erdős)

There exists a constant c such that any family of segments in the plane, no two of which share an endpoint and no three pairwise cross, can be decomposed into at most c subfamilies consisting of pairwise disjoint segments.

Conjecture (Fox and Pach '08)

For every k, there exists a c_k such that any family of convex bodies in the plane, no k of which are pairwise intersecting, can be decomposed into at most c_k subfamilies consisting of pairwise disjoint bodies.

Proof of the main theorem

We will prove something slightly stronger.

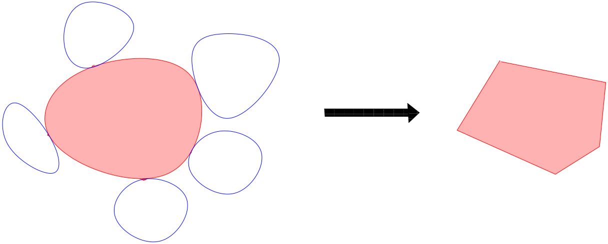
Theorem

Let $\mathcal{C} = \mathcal{R} \cup \mathcal{B}$ be a family of $n > 5$ convex bodies in the plane, where $\mathcal{R}$ and B are pairwise disjoint families, each consisting of pairwise disjoint bodies. Then the number of tangencies between the members of $R$and the members of B is at most $4n - 16$.

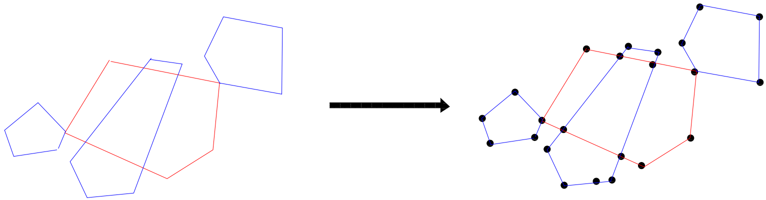
Sketch Proof. Induction on n.

BASE CASE. For $n = 6$, we can have at most 8 tangencies since $K_{3,3}$ is not planar. I.e. we cannot have 3 red pairwise disjoint convex bodies all tangent to 3 blue pairwise disjoint convex bodies.

Now assume the theorem holds for all families of size smaller than n. Let m denote the number of tangencies. Each member in $\mathcal{C}$ is tangent to at least five other members (otherwise we would be done by the induction hypothesis). Replace each member of C by the convex hull of all points of tangencies along its boundary.



Place a vertex at each point of tangency and at each intersection point betweeen the sides of the polygons to obtain the planar graph $G = (V, E)$.



By Euler's polyhedral formula, we have

$$\sum_{v\in V}(d(v)-4)+\sum_{f\in F}(|f|-4)=4(|E|-|V|-|F|)=-8$$

where F is the set of faces in G. Since G is 4-regular, we have

$$\sum_{f\in F}(|f|-4)=-8$$

By defining

- $F(C)$ as the faces inside $C \in \mathcal{C}$,
- $F^{ext} \subset F$ as the faces not inside any member of $\mathcal{C}$,
- F^{int-1} to be the set of faces inside exactly one member of $\mathcal{C}$,

we have

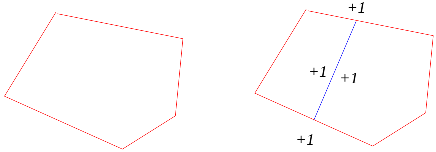
$$\sum_{f \in F^{ext}} (|f| - 4) + \frac{1}{2} \sum_{C \in C} \sum_{f \in F(C)} (|f| - 4) + \frac{1}{2} \sum_{f \in F^{int-1}} (|f| - 4) = -8$$

First Key Observation. If there's no segments in $C \in \mathcal{C}$, we have

$$\sum_{f \in F(C)} (|f| - 4) = |C| - 4.$$

Each segment increases the number of faces by one, and adds four to the total number of sides of these faces. Therefore, for every $C \in \mathcal{C}$ (regardless of the number of segments inside)

$$\sum_{f \in F(C)} (|f| - 4) = |C| - 4.$$



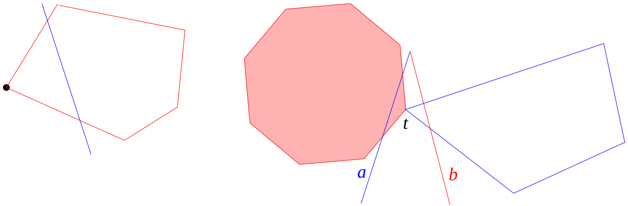
Therefore

$$\frac{1}{2}\sum_{C\in C}\sum_{f\in F(C)}(|f|-4)=\frac{1}{2}\sum_{C\in C}|C|-4=\frac{1}{2}(2m-4n)=m-2n.$$

(Recall that m is the number of tangencies)

handling triangles

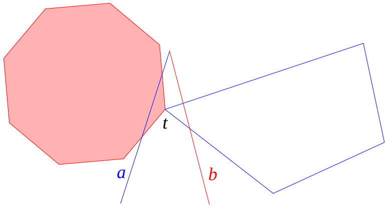
Second Key Observation. All triangles lie inside exactly one convex body at a vertex.



T_3 set of triangle faces and t is the number of vertices adjacent to two triangles (double triangle vertices).

$$|T_3| \leq m + t$$

By convexity, each vertex adjacent to two triangles must be a vertex of an exterior face with at least *six* sides!



Since each exterior face f is incident to at most $|f|/2$ double triangle vertices, we have

$$t \leq \frac{1}{2} \sum_{f \in F_{6+}^{ext}} |f|$$

Therefore we have

$$|T_3| \leq m + \frac{1}{2} \sum_{f \in F_{6+}^{ext}} |f|$$

Since we have a bound on the number of triangles, we have

$$\sum_{f \in F^{int-1}} (|f| - 4) \geq \sum_{f \in T_3} (|f| - 4) = -|T_3| \geq -m - \frac{1}{2} \sum_{f \in F_{6+}^{ext}} |f|$$

Summary

Recall we have by Euler's formula (and by the fact the G is 4-regular)

$$\sum_{f \in F^{ext}} (|f| - 4) + \frac{1}{2} \sum_{C \in C} \sum_{f \in F(C)} (|f| - 4) + \frac{1}{2} \sum_{f \in F^{int-1}} (|f| - 4) = -8$$

Our first key observation

$$\frac{1}{2} \sum_{C \in C} \sum_{f \in F(C)} (|f| - 4) = m - 2n$$

Our second key observation

$$\sum_{f \in F^{int-1}} (|f| - 4) \geq -m - \frac{1}{2} \sum_{f \in F^{ext}_{6+}} |f|$$

Putting it all together

$$\sum_{f\in F^{ext}}(|f|-4)+\frac{1}{2}\sum_{C\in C}\sum_{f\in F(C)}(|f|-4)+\frac{1}{2}\sum_{f\in F^{int-1}}(|f|-4) = -8$$

$$\sum_{f\in F^{ext}_{6+}}(|f|-4)+(m-2n)-\frac{1}{2}\left(m+\frac{1}{2}\sum_{f\in F^{ext}_{6+}}|f|\right) \leq -8$$

$$m/2-2n+\sum_{f\in F^{ext}_{6+}}(\tfrac{3}{4}|f|-4) \leq -8$$

$$m/2-2n \leq -8$$

Hence

$$m \leq 4n-16$$

Regular vertices

Given a collection $\mathcal{C}$ of n convex bodies in the plane. If the boundary of two members of $\mathcal{C}$ intersect at most twice, then we call these intersection points regular (denote $R(\mathcal{C})$). Else they are called irregular (denoted $I(\mathcal{C})$).

Theorem (Ezra, Pach, Sharir)

$$|R(\mathcal{C})| \leq O(n^{4/3+\epsilon})$$

for every $\epsilon > 0$.

Theorem (Pach, Sharir)

$$|R(\mathcal{C})| \leq O(|I(\mathcal{C})| + n)$$

Theorem (Pach, Suk, Treml)

If $\mathcal{C}$ can be decomposed into k subfamilies consisting of pairwise disjoint bodies,

$$|R(\mathcal{C})| \leq O(kn)$$

Other types of regions

Theorem (Pach, Suk, Treml)

The number of tangencies between two families of x-monotone curves in the plane, each consisting of $n > 2$ pairwise disjoint members, is at most $O(n \log^2 n)$.

Theorem (Pinchasi and Ben-Dan)

The number of tangencies between two families of closed Jordan regions in the plane, each consisting of $n > 2$ pairwise disjoint members, is at most $O(n^{3/2} \log n)$.

Last comment on the convex case: By a more careful discharging argument, one can improve the constant a bit (when two triangles are “close” to each other it gives rise to a large 6-face. When there are not a lot of triangles, not every vertex is adjacent to a triangle).
THANK YOU