Tangencies between families of disjoint regions in the plane

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Abstract

Let $\mathcal{C}$ be a family of n convex bodies in the plane, which can be decomposed into k subfamilies of pairwise disjoint sets. It is shown that the number of tangencies between the members of $\mathcal{C}$ is at most $O(kn)$, and that this bound cannot be improved. If we only assume that our sets are connected and *vertically convex*, that is, their intersection with any vertical line is either a segment or the empty set, then the number of tangencies can be *superlinear* in n, but it cannot exceed $O(n \log^2 n)$. Our results imply a new upper bound on the number of regular intersection points on the boundary of $\cup\mathcal{C}$.

1 Introduction

Analyzing the structure of the union of convex bodies or other geometric objects in the plane and in higher dimensions is a classical topic in discrete and computational geometry, with many applications in motion planning and computer graphics (see [AgPS08], for a survey). It was shown in [KLP86] that the number of arcs comprising the boundary of the union of n Jordan regions in the plane, any pair of which share at most two boundary points, is $O(n)$. This fact was applied for planning a collision-free translational motion of a convex robot amidst several polygonal obstacles in the plane. Similar results with algorithmic consequences have been established for "fat" objects [MPS94], [AgS00], [PSS03], [Ez08], [ES07] and "round" objects [ArEK06], [Ef05] in the plane and in higher dimensions. The aim of this paper is to derive a new upper bound on the number of tangencies in an arrangement of convex bodies in the plane.

Two nonoverlapping Jordan *regions* in the plane are said to *touch* each other or to be *tangent* to each other if their boundaries have precisely one point in common and their interiors are disjoint. Two Jordan *curves touch* if they intersect in precisely one point, at which they do not cross each other properly, that is, one curve does not pass from one side of the other curve to the other side.

Estimating the maximum number of tangencies between circles was initiated by de Rocquigny [Ro97] at the end of the 19th century. Erdős's famous unsolved question [Er46] on the maximum number of unit distance pairs among n points in the plane can also be formulated as a problem about tangencies: What is the maximum number tangencies among n (possibly overlapping) disks of unit diameter in the plane? The answer is superlinear in n.

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It was first observed by Tamaki and Tokuyama [TT98] that in order to obtain an upper bound on the number of incidences between a family $\mathcal{C}$ of curves and a set of points in the plane, it is sufficient to estimate the minimum number of points needed to cut the curves in $\mathcal{C}$ into smaller pieces such that any pair of them are either disjoint or cross precisely once. Obviously, this number is at least as large as the number of tangencies between the members of $\mathcal{C}$, and in most cases these two quantities do not differ too much. For a number of applications, this approach leads to the best known upper bounds for the number of incidences between curves and points [AgNP04], [C05], [MT06].

Consider a family $\mathcal{R}$ of n closed Jordan regions in the plane such that any pair of them have *one* or *two* boundary points in common, but no three boundary curves pass through the same point. It was shown in [AgNP04] (see also [AlLP01]) that the number of tangencies between the members of $\mathcal{R}$ is $O(n)$. As is illustrated by Erdős's unit distance problem mentioned above, if we also allow two members of $\mathcal{R}$ to be *disjoint*, then the number of tangencies can be superlinear in n. However, if we count only those tangencies that do not belong to the interior of any member of $\mathcal{R}$, then again we can obtain a linear upper bound [KLP86]. In the last section, we show that our results imply a new upper bound on the number of so called "regular" intersection points along the boundary of the union of all sets in $\mathcal{R}$, which is often better than the best known estimates. This quantity plays a role in analyzing the complexity of higher dimensional arrangements.

In this paper, we study the structure of tangencies between two families of closed Jordan regions, each consisting of n pairwise disjoint members. It was shown by Pinchasi and Ben-Dan [BP07], who independently from us arrived at the same question, that the maximum number of such tangencies is $O(n^{3/2}\log n)$. Their proof is based on a theorem of Marcus-Tardos [MT06] and Pinchasi-Radoičić [PiR04]. They suggested that the correct order of magnitude of the maximum may be linear in n. We start by proving this conjecture in the special case where both families consist of closed *convex* regions (convex *bodies*).

Theorem 1. *The number of tangencies between two families of convex bodies in the plane, each consisting of $n > 2$ pairwise disjoint members, cannot exceed $8n - 16$.*

This bound is not far from being optimal. Figure 1 shows two such families with $(6 - o(1))n$ tangencies. We can start with two slightly rotated hexagons of different colors. Taking n translates of each hexagon and arranging them in a lattice-like fashion as in Figure 1, we obtain two families such that all but $O(\sqrt{n})$ of their members are tangent to *six* hexagons of the opposite color.

Consider a family $\mathcal{C}$ of closed convex bodies in the plane. Assign a vertex to each member of $\mathcal{C}$, and connect two vertices by an edge if the corresponding bodies have a nonempty intersection. The resulting graph $G_{\mathcal{C}}$ is called the *intersection graph* of $\mathcal{C}$. Suppose that the chromatic number of this graph $\chi(G_{\mathcal{C}}) \leq k$, that is, $\mathcal{C}$ can be decomposed into k subfamilies consisting of pairwise disjoint bodies. By denoting n_i as the size of the ith pairwise disjoint subfamily and by Theorem 1, $\mathcal{C}$ has at most

$$\sum_{i=1}^{k-1}\sum_{j>i} O(n_i + n_j) = O(nk)$$

tangencies. Therefore we have the following corollary.

Corollary 2. *Let $\mathcal{C}$ be a family of n convex bodies in the plane, which can be decomposed into k subfamilies consisting of pairwise disjoint bodies. The total number of tangencies between members of $\mathcal{C}$ is $O(kn)$. This bound is tight up to a multiplicative constant.*

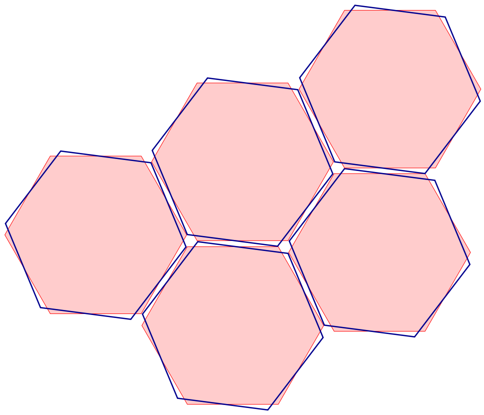

Figure 1: A construction on n pairwise disjoint red convex bodies and n pairwise disjoint blue convex bodies with $6n(1 + o(1))$ tangent pairs.

According to an old conjecture of Erdős, there exists a constant c with the property that any family of segments in the plane, no *two* of which share an endpoint and no *three* are pairwise crossing, can be decomposed into at most c subfamilies consisting of pairwise disjoint segments (see [Ko03], for a survey). More generally, it can be conjectured [FoP08] that there exists a function ϕ such that the chromatic number of the intersection graph of any family of convex bodies with no k pairwise intersecting members is bounded from above by $\phi(k)$. Combining this conjecture with Corollary 2, we would obtain the following statement, which we pose as a conjecture.

Conjecture 3. *For every fixed integer $k > 2$, the number of tangencies in any n-member family of convex bodies, no k of which are pairwise intersecting, is at most $O_k(n)$.*

In Section 3, we address the analogous problem for *vertically convex* sets in the plane, that is, for closed connected sets with the property that every vertical line either misses them or intersects them in a nonempty interval. A curve (connected arc) is *x-monotone* if every vertical line intersects it in at most one point, i.e., if it is vertically convex. Given a vertically convex set r with no vertical boundary interval, the set of upper (lower) endpoints of the segments $\ell \cap r$ for every vertical line ℓ which intersects r forms the *upper (lower) contour* of r. Clearly, the upper and lower contours of r are x-monotone curves. Every tangency between two Jordan regions occurs along their upper or lower contours. Thus, to obtain an upper bound on the number of tangencies between two families of pairwise disjoint vertically convex sets, it is sufficient to estimate the number of tangencies between two families of pairwise disjoint x-monotone curves.

Concerning this problem, we establish the following result in Section 3.

Theorem 4. *Let $f(n)$ denote the maximum number of tangencies between two n-member families of pairwise disjoint x-monotone curves in the plane. Then we have*

$$\Omega(n \log n) \leq f(n) \leq O(n \log^2 n).$$

Corollary 5. *The number of tangencies between two families of vertically convex sets in the plane, each consisting of $n > 2$ pairwise disjoint members, is at most $O(n \log^2 n)$.*

The key step in the proof of Theorem 4 is that we solve the same problem in the special case when one of the families form a "flag".

A family of pairwise disjoint x-monotone curves in the plane is called a *left-flag (right-flag)* if the left (right) endpoint of each of its members lies on a vertical line l, called the "pole" of the flag. We use the same terminology for families of pairwise disjoint vertically convex sets in the plane if their leftmost (rightmost) points lie on the same vertical line.

Theorem 6. (One flag theorem for curves) *Let $f_1(n)$ denote the maximum number of tangencies between n pairwise disjoint x-monotone curves that form a left-flag and a set of n pairwise disjoint x-monotone curves. Then we have $f_1(n) = \Theta(n \log n)$.*

2 Convex bodies

We prove Theorem 3 in the following slightly stronger form.

Theorem 7. *Let $\mathcal{C} = \mathcal{R} \cup \mathcal{B}$ be a family of $n > 5$ convex bodies in the plane, where $\mathcal{R}$ and $\mathcal{B}$ are pairwise disjoint families, each consisting of pairwise disjoint bodies. Then the number of tangencies between the members of $\mathcal{R}$ and the members of $\mathcal{B}$ is at most $4n - 16$.*

Proof of Theorem 7. We will refer to the members of $\mathcal{R}$ and $\mathcal{B}$ as red and blue sets, respectively. We assume, for the sake of simplicity, that no three points of tangency along the boundary of any member of $\mathcal{C}$ are collinear.

The proof is by induction on n. In the base case $n = 6$, the statement readily follows from the fact that a planar graph cannot contain $K_{3,3}$ as a subgraph. Hence for $n - 6$ the maximum number of tangencies is at most 8 and the statement for holds. Suppose now that $n > 6$ and that the theorem has already been verified for all families of size smaller than n. Clearly, we can assume that every member $C \in \mathcal{C}$ is tangent to at least *five* other members. Otherwise, we can delete C and apply the induction hypothesis to the remaining family.

Let m denote the number of red-blue tangencies in $\mathcal{C}$. We start by replacing each member of $\mathcal{C}$ by the convex hull of all points of tangencies along its boundary. That is, we assume that each member of $\mathcal{C}$ is a convex polygon with at least *five* sides, and all tangencies occur at the vertices of these polygons. If we place a vertex at each point of tangency and at each intersection point between the sides of the polygons, then we obtain a *4-regular planar graph* $G = (V, E)$. We can also assume without loss of generality that G is *2-connected*. Let $F = F^{int} \cup F^{ext}$ denote the set of *faces* of G, where F^{int} is the set of *interior* faces that lie inside some convex region in $\mathcal{C}$, and F^{ext} is the set of *exterior* faces that do not lie inside any member of $\mathcal{C}$. Furthermore, let $F^{int-1} \subseteq F^{int}$ be the set of faces that lie inside exactly *one* member of $\mathcal{C}$. For any $C \in \mathcal{C}$, let $|C|$ denote the number of tangencies along the boundary of C, that is, the number of *sides* of the polygon replacing it. Analogously, for any face $f \in F$, let $|f|$ stand for the number of *sides (edges)* of f. Finally, let $F(C) \subseteq F^{int}$ stand for the set of faces that lie inside C.

We would like to use the following form of Euler's polyhedral formula for G.

$$\sum_{v \in V} (d(v) - 4) + \sum_{f \in F} (|f| - 4) = 4(|E| - |V| - |F|) = -8. \tag{1}$$

For any $C \in \mathcal{C}$, the interior of C contains a number of disjoint segments (edges of G) connecting pairs of interior points of the sides of C. Each such edge increases the number of faces within C by *one*, and each adds *four* to the total number of sides of these faces. Therefore, we have

$$\sum_{f \in F(C)} (|f| - 4) = |C| - 4.$$

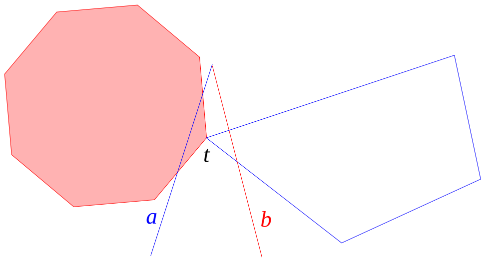

Figure 2: A point from T

This implies

$$\sum_{C \in \mathcal{C}} \sum_{f \in F(C)} (|f| - 4) = \sum_{C \in \mathcal{C}} (|C| - 4) = 2m - 4n. \tag{2}$$

Since each polygon $C \in \mathcal{C}$ has at least *five* sides, triangular interior faces can only occur near the vertices of the polygons C, where a side of some other polygon C' crosses two adjacent sides of C. Also notice that there are no triangular faces in the exterior since $\mathcal{R}$ and $\mathcal{B}$ are pairwise disjoint families. Let $F_3 = \{f \in F : |f| = 3\}$ and let $T \subset V$ be the set of vertices that belong to *two* members of F_3. Notice that each vertex $t \in T$ belongs to two triangular interior faces and to two exterior faces, because G is 2-connected.

Consider the two triangular faces meeting at a vertex $t \in T$, and let a and b denote the sides of the polygons in $\mathcal{C}$ containing the sides of these triangles opposite to t. See Figure 2. These two segments must belong to polygons of different colors. By convexity, a and b cannot have a common point in both exterior faces incident to t. Thus, at least one of the exterior faces incident to t has at least *six* sides. It is also clear that neither of the vertices next to t along this exterior face is a point of tangency between a red and a blue polygon. Therefore, each exterior face f contains at most $|f|/2$ vertices belonging to T. By defining $F^{ext}_{6+} = \{f \in F^{ext} : |f| \geq 6\}$, we have

$$|T| \leq \frac{1}{2} \sum_{f \in F^{ext}_{6+}} |f|,$$

which implies that

$$|F_3| \leq m + |T| \leq m + \frac{1}{2} \sum_{f \in F^{ext}_{6+}} |f|.$$

Therefore, we have

$$\sum_{f \in F^{int-1}} (|f| - 4) \geq \sum_{f \in F_3} (|f| - 4) = -|F_3| \geq -m - \frac{1}{2} \sum_{f \in F^{ext}_{6+}} |f|. \tag{3}$$

Combining (2) and (3), we obtain

$\sum_{f \in F} (|f| - 4)$

$$
\begin{aligned}
&= \sum_{f\in F^{ext}}(|f|-4)+\tfrac{1}{2}\sum_{C\in\mathcal{C}}\sum_{f\in F(C)}(|f|-4)+\tfrac{1}{2}\sum_{f\in F^{int-1}}(|f|-4)\\
&\geq \sum_{f\in F^{ext}_{6+}}(|f|-4)+\tfrac{1}{2}(2m-4n)-\tfrac{1}{2}\left(m+\tfrac{1}{2}\sum_{f\in F^{ext}_{6+}}|f|\right)\\
&= \tfrac{1}{2}m-2n+\sum_{f\in F^{ext}_{6+}}\left(\tfrac{3}{4}|f|-4\right)\\
&\geq \tfrac{1}{2}m-2n.
\end{aligned}
$$

On the other hand, using the fact that G is a 4-regular graph, Euler's formula (1) yields that

$$\sum_{f\in F}(|f|-4)=-8.$$

Comparing the last two relations, we obtain

$$\frac{1}{2}m-2n\leq -8,$$

which implies that $m\leq 4n-16$, as required. □

3 x-monotone curves

In this section, we prove Theorems 4 and 6. We make no attempt here to optimize the constants hidden in the Ω- and O-notation. In the proofs, for simplicity, we omit all floor and ceiling signs whenever these are not crucial. All logarithms are of base 2.

We start with the proof of Theorem 6, for which we need two simple lemmas.

Lemma 8. *Let $\mathcal{R}=\{r_1,...,r_n\}$ and $\mathcal{B}=\{b_1,...,b_n\}$ be a family of n pairwise disjoint x-monotone red curves and a family n pairwise disjoint x-monotone blue curves in the plane such that all of them meet a vertical line l. Then the number of red-blue pairs of curves tangent to each other is at most $8n$.*

Proof. Suppose without loss of generality that no pair of curves (r_i,b_j) touch each other at a point on l. Consider first the right half-plane bounded by l and only the point of tangency for which a red curve touches a blue curve from *below*. Among these tangencies, the rightmost point of tangency along the curve r_i or b_i is called *extreme*. The number of pairs (r_i,b_j) for which r_i touches b_j from below is at most $2n$. Indeed, there are altogether at most $2n$ extreme tangencies, and there are no non-extreme point of tangency since the curves are x-monotone. See Figure 3. Analogously, the number of pairs (r_i,b_j) for which r_i touches b_j from *above* in the right half-plane is at most $2n$. A symmetric argument shows that the number of pairs (r_i,b_j) that touch each other in the left half-plane bounded by l is also at most $4n$. □

Lemma 9. *Let $\mathcal{R}=\{r_1,...,r_n\}$ and $\mathcal{B}=\{b_1,...,b_n\}$ be a family of pairwise disjoint x-monotone red curves and a family of pairwise disjoint x-monotone blue curves in the plane. Suppose that there are two vertical lines, l_1 and l_2, which intersect every $b_i\in\mathcal{B}$. Then the number of pairs (r_i,b_j) that touch each other in the strip between l_1 and l_2 is at most $2n$.*

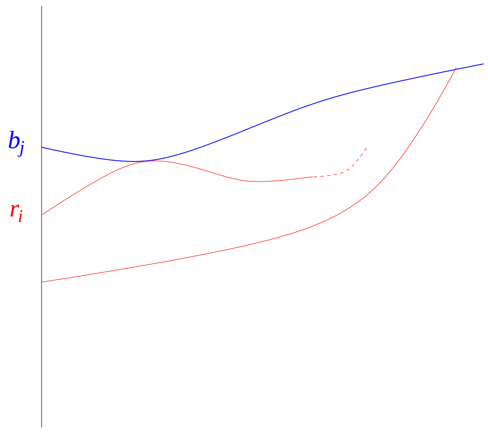

Figure 3: If the point where r_i touches b_j from below is not an extreme tangency, then b_j must touch some red curve below r_i, and r_i must touch some blue curve above b_j. However this is impossible since the curves are x-monotone.

Proof. Each red curve can be tangent to at most *two* blue curves. □

We are now ready to prove the upper bound of Theorem 6 in the following slightly stronger form.

Theorem 10. *Let $\mathcal{C} = \mathcal{R} \cup \mathcal{B}$ be a family of n x-monotone curves such that $\mathcal{R}$ consists of pairwise disjoint red curves with their left endpoints on a vertical line l (left-flag), and $\mathcal{B}$ is a family of pairwise disjoint blue curves lying entirely to the right of l. Then the number of red-blue tangencies between the curves is at most $30n \log n$.*

Proof. Without loss generality, we can assume that x-coordinate of the endpoints of each curve are distinct. We proceed by induction on n. The base case is trivial. For the inductive step, the proof falls into two cases.

Case 1. Assume $|\mathcal{B}| \geq n/2$. Then there exists a vertical line l', such that there are exactly $|\mathcal{B}|/2$ blue curves completely to the right of l'. Now let

$$\mathcal{R}_1 = \{r \in \mathcal{R} : r \text{ lies completely to the left of } l'\}$$

$$\mathcal{R}_2 = \{r \in \mathcal{R} : r \text{ intersects } l'\}$$

and

$$\mathcal{B}_1 = \{b \in \mathcal{B} : b \text{ lies completely to the left of } l' \text{ or intersects } l'\}$$

$$\mathcal{B}_2 = \{b \in \mathcal{B} : b \text{ lies completely to the right of } l'\}.$$

By Lemma 8 and 9, the number of tangent pairs between $\mathcal{R}_2$ and $\mathcal{B}_1$ is at most $10n$. Since $|\mathcal{R}_1 \cup \mathcal{B}_1|$ and $|\mathcal{R}_2 \cup \mathcal{B}_2|$ is at most $3n/4$, by the induction hypothesis, the number of tangent pairs is at most

$$\begin{aligned} &\leq 10n + 30|\mathcal{R}_1 \cup \mathcal{B}_1| \log(3n/4) + 30|\mathcal{R}_2 \cup \mathcal{B}_2| \log(3n/4) \\ &= 30n \log n + 10n - 30n \log(4/3) \\ &\leq 30n \log n. \end{aligned}$$

Case 2. If $|\mathcal{R}| \geq n/2$, then there exists a vertical line l' such that there are exactly $|\mathcal{R}|/2$ red curves completely to the left of l'. By following the exact same argument as in Case 1, the number of tangent pairs is at most $30n \log n$.

□

Proof of Theorem 6. The upper bound follows from Theorem 10. The lower bound follows from the following construction of n-member red left-flag and an n-member right-flag with $\Omega(n \log n)$ tangencies between them. The construction is recursive. Let $\{r_1, \ldots, r_{n/2}\}$ and $\{b_1, \ldots, b_{n/2}\}$ be a red left-flag and a blue right-flag with poles $x = 2$ and $x = 3$, respectively, with $f_1(n/2)$ tangencies between them. Let $\{r_{n/2+1}, \ldots, r_n\}$ and $\{b_{n/2+1}, \ldots, b_n\}$ be a red left-flag and a blue right-flag with poles $x = 0$ and $x = 1$, having $f_1(n/2)$ tangencies. It is easy to see that the curves r_i $(i \leq n/2)$ can be extended to the left and the curves b_j $(j > n/2)$ to the right until they hit the lines $x = 0$ and $x = 3$, respectively, so that r_i touches from above $b_{i+n/2}$, for every $i \leq n/2$, in the vertical strip $1 < x < 2$. See Fig. 4. Hence, the maximum number of tangencies between an n-member left-flag and n-member right-flag satisfies the recurrence

$$f_1(n) \geq 2f_1(n/2) + n/2,$$

which implies that $f_1(n) = \Omega(n \log n)$.

□

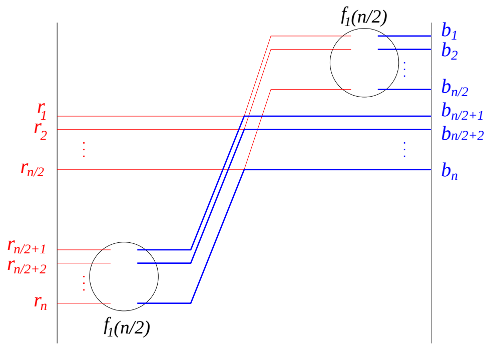

Figure 4: The construction.

The lower bound in Theorem 4 follows from Theorem 6. The upper bound immediately follows from the following lemma.

Lemma 11. *Let $\mathcal{C} = \mathcal{R} \cup \mathcal{B}$ be an n-member family of x-monotone curves such that both $\mathcal{R}$ and $\mathcal{B}$ consist of pairwise disjoint curves in the plane. Then the number of tangent pairs in $\mathcal{R} \cup \mathcal{B}$ is at most $100n \log^2 n$.*

Proof. Induction on n. The base cases $n = 1, 2, 3, 4$ are trivial. For the inductive step, we can assume the x-coordinate of the endpoints of all n curves are distinct. Hence, there exists a vertical line l, such that there are exactly $n/2$ curves completely to the left of l, and at most $n/2$ curves completely to the right of l. Let

$$\mathcal{R}_1 = \{r \in \mathcal{R} : r \text{ lies completely to the left of } l\}$$

$$\mathcal{R}_2 = \mathcal{R} \setminus \mathcal{R}_1$$

and likewise

$$\mathcal{B}_1 = \{b \in \mathcal{B} : b \text{ lies completely to the left of } l\}$$

$$\mathcal{B}_2 = \mathcal{B} \setminus \mathcal{B}_1.$$

By Theorem 10, the number of tangent pairs in $\mathcal{R}_1 \cup \mathcal{B}_2$ and in $\mathcal{R}_2 \cup \mathcal{B}_1$ is at most $30n \log n$. Since $|\mathcal{R}_1 \cup \mathcal{B}_1| = |\mathcal{R}_2 \cup \mathcal{B}_2| = n/2$ and $n \geq 5$, the maximum number of tangencies is at most

$$2 \cdot 30n \log n + 2 \cdot 100(n/2) \log^2(n/2)) \leq 100n \log^2 n.$$

□

4 Concluding remarks

Given a family $\mathcal{C}$ of n convex bodies in the plane, no three of which share a boundary point, it was shown by Erdős and Grünbaum [ErG73] that the number of tangencies not contained in the interior of a third region can be as large as $\Omega(n^{4/3})$. It was proved in [EPS09] that this bound is not far from being optimal.

More generally, if the boundaries of two members of $\mathcal{C}$ intersect at most twice, then we call these intersection points *regular*. All other boundary intersections are called *irregular*. Let $R(\mathcal{C})$ and $I(\mathcal{C})$ denote the sets of regular and irregular intersection points that belong to the boundary of the union of all members of $\mathcal{C}$. It readily follows from the result in [EPS09], referred to in the last paragraph, that

$$|R(\mathcal{C})| = O(n^{4/3+\varepsilon}),$$

for every $\varepsilon > 0$. Pach and Sharir [PS99] established another upper bound on the number of regular intersection points on the boundary of $\cup\mathcal{C}$:

$$|R(\mathcal{C})| = O(|I(\mathcal{C})| + n). \tag{4}$$

Using, e.g., Lemma 1 in [ArEH01], it is easy to reformulate Corollary 2, as follows.

Theorem 12. *Let $\mathcal{C}$ be a family of n convex bodies in the plane, no three of which share a boundary point. Suppose that $\mathcal{C}$ can be decomposed into k subfamilies consisting of pairwise disjoint bodies. Then we have*

$$|R(\mathcal{C})| = O(kn).$$

Note that this bound is often better than the previous two estimates. For instance, for the family depicted on Fig. 5, $|I(\mathcal{C})| = \Theta(n^2)$, so that (4) implies a quadratic upper bound on $|R(\mathcal{C})|$. On the other hand, Theorem 12 gives a linear bound, as the condition is satisfied with $k = 2$.

Two families $\mathcal{R}$ and $\mathcal{B}$ of convex bodies are said to be *touching* if every member of $\mathcal{R}$ is tangent to all members of $\mathcal{B}$. It was shown in [KaP94] that there exists a constant $c > 0$ such that for any pair $\mathcal{R}, \mathcal{B}$ of touching n-member families of convex bodies in the plane, $\mathcal{R}$ or $\mathcal{B}$ must have at least cn members that share a point. If $\mathcal{R}$ and $\mathcal{B}$ are touching families with $|\mathcal{R}|, |\mathcal{B}| \geq 6$, then it is conjectured that at least one of them has *three* members that share a point.

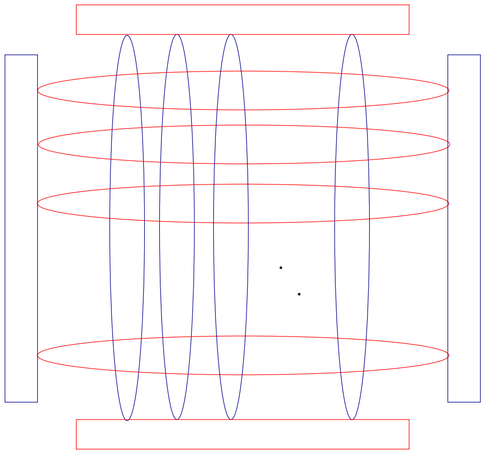

Figure 5: $n/2$ horizontal red ellipses crossing $n/2$ vertical blue ellipses such that each ellipse touches precisely two rectangles.

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