Test 1

MCS 421 Combinatorics

Problem 1. Prove the Erdős-Szekeres monotone subsequence theorem: Given a sequence $a_1, a_2, \ldots$ of $(n-1)^2+1$ distinct real numbers, there is a subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$, where $i_1 < i_2 < \cdots < i_n$, that is either increasing or decreasing.

Solution: Page 76, Application 9 in Chapter 3.

Problem 2. Prove that $r(K_3, K_3) = 6$. That is, prove that $r(K_3, K_3) \leq 6$, and show that $r(K_3, K_3) > 5$ by coloring the edges of K_5 with two colors such that it does not contain a monochromatic triangle.

Solution: Start with a vertex v, and consider the 5 edges emanating out of it. By pigeonhole, there is a color, say red, that appears three times. Let x, y, z be the vertices at the end of these red edges. Now if any edge between x, y, z is red, then we have a red triangle. If they are all blue, then we have a blue triangle. QED.

By taking a red cycle, and a blue cycle of length 5, we can color the edges of K_5 with two colors, and no monochromatic triangle.

Problem 3. 20 identical sticks are lined up in a row occupying 20 distinct places.

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Three of them are to be chosen. How many choices are there if there must be at least two sticks between each pair of chosen sticks?

Solution: Let x_1, x_2, x_3 be the location (between 1 and 20) of the three sticks chosen. Let y_1, y_2, y_3, y_4 be non negative integers such that y_i denote the number of sticks between x_{i-1} and x_i. Hence $y_1 \geq 0$, $y_2 \geq 2$, $y_3 \geq 2$, and $y_4 \geq 0$ where $y_1 + y_2 + y_3 + y_4 = 17$. By substituting $z_2 = y_2 - 2$ and $z_3 = y_3 - 2$, we need to solve

$$y_1 + z_2 + z_3 + y_4 = 13,$$

where $y_1, z_2, z_3, y_4 \geq 0$. Hence the solution is $\binom{16}{3}$.

Problem 4. How many odd numbers between 1,000,000 and 9,999,999 have distinct digits?

Solution: (5)(8)(8)(7)(6)(5)(4).

Problem 5. Find the coefficient of x^4y^2 in the expansion of $(-2x + 4y)^6$.

Solution: By the binomial theorem, the coefficient is $(-2)^4(y)^2\binom{6}{2}$.

Problem 6. Prove that if 11 integers are selected from among $\{1, 2, ..., 20\}$, then the selection includes integers a and b such that $a - b = 2$.

Solution: Partition the integers into sets

$$\{1,3\}, \{2,4\}, \{5,7\}, \{6,8\}, \{9,11\}, \{10,12\}, \{13,15\}, \{14,16\}, \{17,19\}, \{18,20\}.$$

By pigeonhole, there are two integers from the same group, which implies that there is a pair with difference 2.