Test 1, MCS 421

Problem 1. Prove (from scratch) the Erdős-Szekeres monotone subsequence theorem: Every sequence $a_1, a_2, \ldots, a_{n^2+1}$ of n^2+1 distinct real numbers contains either an increasing subsequence of length $n+1$ or a decreasing subsequence of length $n+1$.

Solution: Page 76, Application 9 in Chapter 3.

Problem 2. What is the coefficient of $x_1^3 x_2^3 x_3 x_4^2$ in the expansion of $(x_1 - x_2 + 2x_3 - 2x_4)^9$.

Solution: $\frac{9!}{3!3!2!}(-1)^3(2)(-2)^2$.

Problem 3. Let $X = \{1, 2, \ldots, 12\}$ be the set of the first 12 positive integers, and let $\mid$ be the relation on the pairs of X, where for $x, y \in X$, $x \mid y$ if x divides y. For example, $2 \mid 4$, but $2 \nmid 3$. It is easy to see that $(X, \mid)$ is a poset. Determine a chain of largest size in X, and partition X into the smallest number of antichains.

Solution: Largest chain is $\{1, 2, 4, 8\}$. Decomposition into four antichains:

$$\{1\}, \{2, 3, 5, 7\}, \{4, 6, 9, 10, 11\}, \{8, 12\}.$$

Problem 4. There are $2n+1$ identical books to be put in a bookcase with three shelves. In how many ways can this be done if each pair of shelves together contains more books than the other shelf?

Solution: Let x_i denote the number of books on shelf i. Then we must have $x_i \leq n$, and we want to count the number of integer solutions $x_1 + x_2 + x_3 = 2n+1$. Set $y_i = n - x_i \geq 0$, which implies $y_1 + y_2 + y_3 = n - 1$. There is a one-to-one relationship between the integer solutions for the x_i and the integer solutions for the y_i. Hence it suffices to solve the number of solutions for the y_i. By stars and bars, we have $\binom{n-1+2}{2} = \binom{n+1}{2}$.

Problem 5. We are to seat five boys, five girls, and one parent in a circular arrangement around a table. In how many ways can this be done if no boy is to sit next a boy and no girl is to sit next to a girl?

Solution: There is one way to seat the parent down first. Starting from the parent's right, we have boy-girl-boy-girl-...... or girl-boy-girl-boy-.... For each scenario, there are 5! ways to sit the boys and 5! ways to sit the girls, which gives us a total of $2(5!)^2$ solutions.

Problem 6. Prove that, for any $n+1$ integers $a_1, a_2, \ldots, a_{n+1}$, there exist two of the integers a_i and a_j with $i \neq j$ such that $a_i - a_j$ is divisible by n.

Solution: Dividing n into a_i gives a remainder of $0, 1, 2, \ldots,$ or $n-1$. By pigeonhole, there are two numbers a_i and a_j whose remainder are the same. Hence $a_i - a_j$ is divisible by n.