Math 310, Test 1 Solutions, 10 February 2017

Problem 1. Determine all values h and k such that the system $x_1+3x_2=k$, $4x_1+hx_2=8$ has i) no solutions, ii) contains a unique solution, iii) has infinitely many solutions.

Solution. We have the augmented matrix $\begin{pmatrix} 1 & 3 & k \\ 4 & h & 8 \end{pmatrix}$. After performing elementary row operations, we have the echelon form matrix $\begin{pmatrix} 1 & 3 & k \\ 0 & h-12 & 8-4k \end{pmatrix}$. i) We have no solutions when $h=12$ and $k \neq 2$. ii) We have a unique solution when $h \neq 12$. iii) we have infinitely many solutions when $h=12$ and $k=2$.

Problem 2. Find the LU factorization of $\begin{pmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{pmatrix}$.

Solution. $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ -3 & -3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -4 & -2 & 3 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Problem 3. Let $T: R^2 \to R^3$ be a linear transformation such that $T(x_1,x_2) = (x_1 - 2x_2, -x_1+3x_2, 3x_1-2x_2)$. Find (b_1,b_2) such that $T(b_1,b_2) = (-1,4,9)$. Is T one-to-one? Is T onto? Is T invertible?

Solution. We have the transformation matrix $A = \begin{pmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{pmatrix}$. After performing elementary row operations, we have the echelon form matrix $\begin{pmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. Since there is a pivot in every column, T is one-to-one. Since there is NOT a pivot in every row, T is not onto. T is not invertible since A is not a square matrix.

Problem 4. Describe the solution set of the system $3x_1+5x_2-4x_3=0, -3x_1-2x_2+4x_3=0, 6x_1+x_2-8x_3=0$ using parametric vector form.

Solution. We have the augmented matrix $\begin{pmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{pmatrix}$. After performing elementary row operations, we have the echelon form matrix $\begin{pmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The x_3 is a free variable, $x_2 = 0$, and $x_1 = 4x_3/3$. Our solution in parametric vector form is $x_3 \begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix}$, where x_3 is any real number.

Problem 5. Let $v_1 = \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}$, $v_2 = \begin{pmatrix} -2 \\ 10 \\ 6 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ -9 \\ h \end{pmatrix}$. For what values of h is v_3 in $Span\{v_1, v_2\}$? For what values of h is $\{v_1, v_2, v_3\}$ linear dependent?

Solution. v_3 is in the Span when there are solutions to the system whose augmented matrix is $\begin{pmatrix} 1 & -2 & 2 \\ -5 & 10 & -9 \\ -3 & 6 & h \end{pmatrix}$. After performing elementary row operations, we have the matrix $\begin{pmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ -3 & 6 & h \end{pmatrix}$. The second row says that $0 = 1$ which is a contradiction. Hence v_3 is NOT in $Span\{v_1, v_2\}$ for all values of h. The vectors v_1, v_2, v_3 are dependent for all values of h since $-2v_1 = v_2$.

Problem 6. Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $v_1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$, and let $T : R^2 \to R^2$ be a linear transformation that maps x into $x_1v_1 + x_2v_2$. Find a matrix A such that $T(x) = Ax$ for each x. Is T invertible? If so, find A^{-1}.

Solution. $A = \begin{pmatrix} -2 & 7 \\ 5 & -3 \end{pmatrix}$. Since $(-2)(-3) - (7)(5) \neq 0$, A is invertible, and $A^{-1} = (1/-29) \begin{pmatrix} -3 & -7 \\ -5 & -2 \end{pmatrix}$