Midterm 2, 5 April 2017

Problem 1. Let W be a subspace of $\mathbb{R}^3$, and let $W^\perp$ be the orthogonal complement of W. Prove or disprove that $W^\perp$ is a subspace.

Solution. 1) the 0 vector is in $W^\perp$ since $0 \cdot w = 0$ for any w in W. 2) For any two vectors u, v in $W^\perp$, notice that

$$(u + v) \cdot w = u \cdot w + v \cdot w = 0 + 0 = 0$$

for any w in W. Hence $(u+v)$ is in $W^\perp$. 3) Finally for v in $W^\perp$, $(\alpha v)\cdot w = \alpha(v\cdot w) = \alpha(0) = 0$. So (αv) is also in $W^\perp$. Therefore, $W^\perp$ is a subspace.

Problem 2. Suppose A is row equivalent to B where. Find $rank(A)$ and $dim(Nul(A))$. Then find bases for $Col(A)$, $Row(A)$, and $Nul(A)$.

$$A = \begin{pmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 1 & 2 & -4 & 10 & 13 & -12 \\ 1 & -1 & -1 & 1 & 1 & -3 \\ 1 & -3 & 1 & -5 & -7 & 3 \\ 1 & -2 & 0 & 0 & -5 & -4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 & -3 & 7 & 9 & -9 \\ 0 & 1 & -1 & 3 & 4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution. $Rank(A) = 3$, and the basis for $Col(A)$ is $\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ -3 \\ -2 \end{pmatrix}, \begin{pmatrix} 7 \\ 10 \\ 1 \\ -5 \\ 0 \end{pmatrix} \}$. The dimension of $Row(A)$ is 3, and its basis is the first three rows of matrix B. The dimension of $Null(A)$ is 3. Back solving the homogeneous system given by matrix B gives the basis $\{ \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \}$.

Problem 3. Let $\mathcal{A} = \{a_1, a_2, a_3\}$ and $\mathcal{B} = \{b_1, b_2, b_3\}$ be bases for a vector space V, and suppose $b_1 = 2a_1 - a_2 + a_3$, $b_2 = 3a_2 + a_3$, and $b_3 = -3a_1 + 2a_3$. Find the change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{A}$. Find $[x]_\mathcal{A}$ for $x = b_1 - 2b_2 + 2b_3$.

Solution. The change of coordinate matrix is $\begin{pmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}$. Since $x = b_1 - 2b_2 + 2b_3 = -4a_1 - 7a_2 + 3a_3$, we have $[x]_\mathcal{A} = \begin{pmatrix} -4 \\ -7 \\ 3 \end{pmatrix}$.

Problem 4. Diagonalize the matrix $A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$. That is write $A = PDP^{-1}$ where D is a diagonal matrix, and P is an invertible matrix. Then compute A^{12}.

Solution. We have the characteristic equation $(2-\lambda)(1-\lambda) - 12 = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2) = 0$. So $\lambda_1 = 5, \lambda_2 = -2$. For $\lambda_1 = 5$, we solve the homogeneous system $\begin{pmatrix} -3 & 3 & 0 \\ 4 & -4 & 0 \end{pmatrix}$. This gives the general solution $x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For $\lambda_2 = -2$, we solve the homogeneous system $\begin{pmatrix} 4 & 3 & 0 \\ 4 & 3 & 0 \end{pmatrix}$. This gives the general solution $x_1 \begin{pmatrix} -3/4 \\ 1 \end{pmatrix}$, so $v_2 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$. Therefore, $P = \begin{pmatrix} 1 & -3 \\ 1 & 4 \end{pmatrix}$ and $P^{-1} = (1/7)\begin{pmatrix} 4 & 3 \\ -1 & 1 \end{pmatrix}$. Hence

$$A = \begin{pmatrix} 1 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix} (1/7) \begin{pmatrix} 4 & 3 \\ -1 & 1 \end{pmatrix}.$$

and

$$A^{12} = \begin{pmatrix} 1 & -3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5^{12} & 0 \\ 0 & -2^{12} \end{pmatrix} (1/7) \begin{pmatrix} 4 & 3 \\ -1 & 1 \end{pmatrix}.$$

Problem 5. Let $A = \begin{pmatrix} 5 & -2 \\ 1 & 3 \end{pmatrix}$ be a matrix acting on $\mathbb{C}^2$. Find the eigenvalues and a basis for each eigenspace in $\mathbb{C}^2$. Then find an invertible matrix P and a matrix C of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ such that $A = PCP^{-1}$.

Solution. We have the characteristic equation $(5-\lambda)(3-\lambda) + 2 = \lambda^2 - 8\lambda + 17$ Using the quadratic formula, we have $\lambda_{1,2} = \frac{8 \pm \sqrt{-4}}{2} = 4 \pm i$. Set $\lambda = 4 - i$. Then $C = \begin{pmatrix} 4 & -1 \\ 1 & 4 \end{pmatrix}$. To find the eigenvectors, we solve the homogeneous system $\begin{pmatrix} 1+i & -2 & 0 \\ 1 & -1+i & 0 \end{pmatrix}$. back solving gives us the general solution $x_2 \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$.

Therefore, $P = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. Hence $A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

Problem 6. Let $H \subset \mathbb{P}_3$ be the set of polynomials of the form $p(t) = at^3 + 2$, where a is a real number. Is H a subspace? Why or why not?

Solution. For all values of a, the zero polynomial is not obtained. Since the zero polynomial is not in H, it is not a subspace.