Exercises

1. Let S and T be $(d+1)$-point sets in $\mathbf{R}^d$, each containing 0 in the convex hull. Prove that there exists a finite sequence $S_0 = S, S_1, S_2, \ldots, S_m = T$ of $(d+1)$-point sets with $S_i \subseteq S \cup T$ and $0 \in \operatorname{conv}(S_i)$ for all i, such that S_{i+1} is obtained from S_i by deleting one point and adding another. Assume general position of $S \cup T$ if convenient. Warning: better do not try to find a $(d+1)$-term sequence. $\boxed{3}$

8.3 Tverberg's Theorem

Radon's lemma states that any set of $d+2$ points in $\mathbf{R}^d$ has two disjoint subsets whose convex hulls intersect. Tverberg's theorem is a generalization of this statement, where we want not only two disjoint subsets with intersecting convex hulls but r of them.

It is not too difficult to show that if we have very many points, then such r subsets can be found. For easier formulations, let $T(d,r)$ denote the smallest integer T such that for any set A of T points in $\mathbf{R}^d$ there exist pairwise disjoint subsets $A_1, A_2, \ldots, A_r \subset A$ with $\bigcap_{i=1}^r \operatorname{conv}(A_i) \neq \emptyset$. Radon's lemma asserts that $T(d,2) = d+2$.

It is not hard to see that $T(d, r_1 r_2) \leq T(d,r_1)T(d,r_2)$ (Exercise 1). Together with Radon's lemma this observation shows that $T(d,r)$ is finite for all r, but it does not give a very good bound.

Here is another, more sophisticated, argument, leading to the (still suboptimal) bound $T(d,r) \leq n = (r-1)(d+1)^2 + 1$. Let A be an n-point set in $\mathbf{R}^d$ and let us set $s = n - (r-1)(d+1)$. A simple counting shows that every $d+1$ subsets of A of size s all have a point of A in common. Therefore, by Helly's theorem, the convex hulls of all s-tuples have a common point x (typically not in A anymore). By Carathédory's theorem, x is contained in the convex hull of some $(d+1)$-point set $A_1 \subseteq A$. Since $A \setminus A_1$ has at least s points, x is still contained in $\operatorname{conv}(A \setminus A_1)$, and thus also in the convex hull of some $(d+1)$-point $A_2 \subseteq A \setminus A_1$, etc. We can continue in this manner and select the desired r disjoint sets $A_1, \ldots, A_r$, all of them containing x in their convex hulls.

It is not difficult to see that $T(d,r)$ cannot be smaller than $(r-1)(d+1)+1$ (Exercise 2). Tverberg's theorem asserts that this smallest conceivable value is always sufficient.

8.3.1 Theorem (Tverberg's theorem). *Let d and r be given natural numbers. For any set $A \subset \mathbf{R}^d$ of at least $(d+1)(r-1)+1$ points there exist r pairwise disjoint subsets $A_1, A_2, \ldots, A_r \subseteq A$ such that $\bigcap_{i=1}^r \operatorname{conv}(A_i) \neq \emptyset$.*

The sets $A_1, A_2, \ldots, A_r$ as in the theorem are called a *Tverberg partition* of A (we may assume that they form a partition of A), and a point in the intersection of their convex hulls is called a *Tverberg point.* The following illustration shows what such partitions can look like for $d = 2$ and $r = 3$; both the drawings use the same 7-point set A:

(Are these all Tverberg partitions for this set, or are there more?)

As in the colorful Carathéodory theorem, a very interesting open problem is the existence of an efficient algorithm for finding a Tverberg partition of a given set. There is a polynomial-time algorithm if the dimension is fixed, but some NP-hardness results for closely related problems indicate that if the dimension is a part of input then the problem might be algorithmically difficult.

Several proofs of Tverberg's theorem are known. The one demonstrated below is maybe not the simplest, but it shows an interesting "lifting" technique. We deduce the theorem by applying the colorful Carathéodory theorem to a suitable point configuration in a higher-dimensional space.

Proof of Tverberg's theorem. We begin with a reformulation of Tverberg's theorem that is technically easier to handle. For a set $X \subseteq \mathbf{R}^d$, the *convex cone generated by* X is defined as the set of all linear combinations of points of X with nonnegative coefficients; that is, we set

$$\operatorname{cone}(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i \colon x_1, \ldots, x_n \in X,\ \alpha_1, \ldots, \alpha_n \in \mathbf{R},\ \alpha_i \geq 0 \right\}.$$

Geometrically, cone(X) is the union of all rays starting at the origin and passing through a point of conv(X). The following statement is equivalent to Tverberg's theorem:

8.3.2 Proposition (Tverberg's theorem: cone version). *Let A be a set of $(d+1)(r-1)+1$ points in $\mathbf{R}^{d+1}$ such that $0 \notin \operatorname{conv}(A)$. Then there exist r pairwise disjoint subsets $A_1, A_2, \ldots, A_r \subseteq A$ such that $\bigcap_{i=1}^{r} \operatorname{cone}(A_i) \neq \{0\}$.*

Let us verify that this proposition implies Tverberg's theorem. Embed $\mathbf{R}^d$ into $\mathbf{R}^{d+1}$ as the hyperplane $x_{d+1} = 1$ (as in Section 1.1). A set $A \subset \mathbf{R}^d$ thus becomes a subset of $\mathbf{R}^{d+1}$; moreover, its convex hull lies in the $x_{d+1} = 1$ hyperplane, and thus it does not contain 0. By Proposition 8.3.2, the set A can be partitioned into groups $A_1, \ldots, A_r$ with $\bigcap_{i=1}^{r} \operatorname{cone}(A_i) \neq \{0\}$. The intersection of these cones thus contains a ray originating at 0. It is easily checked that such a ray intersects the hyperplane $x_{d+1} = 1$ and that the intersection point is a Tverberg point for A. Hence it suffices to prove Proposition 8.3.2.

Proof of Proposition 8.3.2. Let us put $N = (d+1)(r-1)$; thus, A has $N+1$ points. First we define linear maps $\varphi_j: \mathbf{R}^{d+1} \to \mathbf{R}^N$, $j = 1, 2, \ldots, r$. We group the coordinates in the image space $\mathbf{R}^N$ into $r-1$ blocks by $d+1$ coordinates each. For $j = 1, 2, \ldots, r-1$, $\varphi_j(x)$ is the vector having the coordinates of x in the jth block and zeros in the other blocks; symbolically,

$$\varphi_j(x) = \big(\underbrace{0\,|\,0\,|\cdots|\,0}_{(j-1)\times}\,|\,x\,|\,0\,|\cdots|\,0\big).$$

The last mapping, φ_r, has $-x$ in each block: $\varphi_r(x) = (-x\,|-x\,|\cdots|-x\,)$.

These maps have the following property: For any r vectors $u_1, \ldots, u_r \in \mathbf{R}^{d+1}$,

$$\sum_{j=1}^{r} \varphi_j(u_j) = 0 \;\text{ holds if and only if }\; u_1 = u_2 = \cdots = u_r. \tag{8.1}$$

Indeed, this can be easily seen by expressing

$$\sum_{j=1}^{r} \varphi_j(u_j) = (u_1 - u_r\,|\,u_2 - u_r\,|\cdots|\,u_{r-1} - u_r)\,.$$

Next, let $A = \{a_1, \ldots, a_{N+1}\} \subset \mathbf{R}^{d+1}$ be a set with $0 \notin \operatorname{conv}(A)$. We consider the set $M = \varphi_1(A) \cup \varphi_2(A) \cup \cdots \cup \varphi_r(A)$ in $\mathbf{R}^N$ consisting of r copies of A. The first $r-1$ copies are placed into mutually orthogonal coordinate subspaces of $\mathbf{R}^N$. The last copy of each a_i sums up to 0 with the other $r-1$ copies of a_i. Then we color the points of M by $N+1$ colors; all copies of the same a_i get the color i. In other words, we set $M_i = \{\varphi_1(a_i), \varphi_2(a_i), \ldots, \varphi_r(a_i)\}$. As we have noted, the points in each M_i sum up to 0, which means that $0 \in \operatorname{conv}(M_i)$, and thus the assumptions of the colorful Carathéodory theorem hold for $M_1, \ldots, M_{N+1}$.

Let $S \subseteq M$ be a rainbow set (containing one point of each M_i) with $0 \in \operatorname{conv}(S)$. For each i, let $f(i)$ be the index of the point of M_i contained in S; that is, we have $S = \big\{\varphi_{f(1)}(a_1), \varphi_{f(2)}(a_2), \ldots, \varphi_{f(N+1)}(a_{N+1})\big\}$. Then $0 \in \operatorname{conv}(S)$ means that

$$\sum_{i=1}^{N+1} \alpha_i \varphi_{f(i)}(a_i) = 0$$

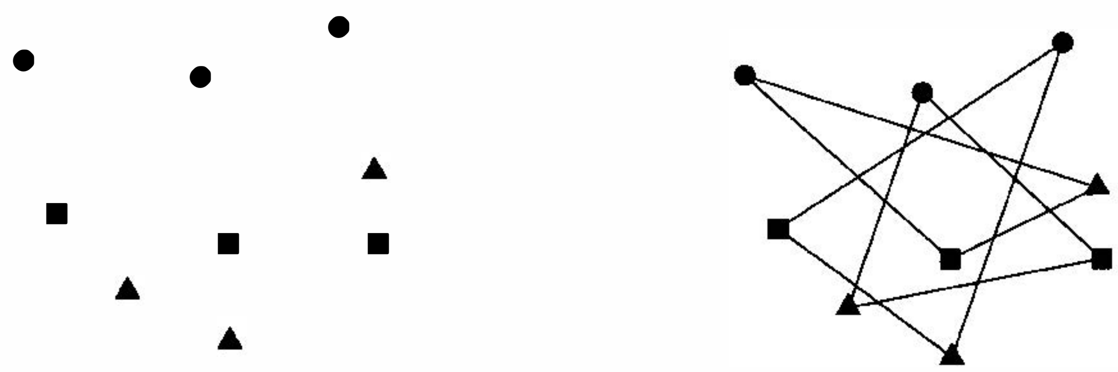
for some nonnegative real numbers $\alpha_1, \ldots, \alpha_{N+1}$ summing to 1. Let I_j be the set of indices i with $f(i) = j$, and set $A_j = \{a_i\colon i \in I_j\}$. The above sum can be rearranged:

$$\sum_{i=1}^{N+1} \alpha_i \varphi_{f(i)}(a_i) = \sum_{j=1}^{r} \sum_{i \in I_j} \alpha_i \varphi_j(a_i) = \sum_{j=1}^{r} \varphi_j\Big(\sum_{i \in I_j} \alpha_i a_i\Big)$$

(the last equality follows from the linearity of each φ_j). Write $u_j = \sum_{i \in I_j} \alpha_i a_i$. This is a linear combination of points of A_j with nonnegative coefficients, and hence $u_j \in \operatorname{cone}(A_j)$. Above we have derived $\sum_{j=1}^r \varphi_j(u_j) = 0$, and so by (8.1) we get $u_1 = u_2 = \cdots = u_r$. Hence the common value of all the u_j belongs to $\bigcap_{j=1}^r \operatorname{cone}(A_j)$.

It remains to check that $u_j \neq 0$. Since we assume $0 \notin \operatorname{conv}(A)$, the only nonnegative linear combination of points of A equal to 0 is the trivial one, with all coefficients 0. On the other hand, since not all the α_i are 0, at least one u_j is expressed as a nontrivial linear combination of points of A. This proves Proposition 8.3.2 and Tverberg's theorem as well. □

The colored Tverberg theorem. If we have 9 points in the plane, 3 of them red, 3 blue, and 3 white, it turns out that we can always partition them into 3 triples in such a way that each triple has one red, one blue, and one white point, and the 3 triangles determined by the triples have a nonempty intersection.



The colored Tverberg theorem is a generalization of this statement for arbitrary d and r. We will need it in Section 9.2, for a result about many simplices with a common point. In that application, the colored version is essential (and Tverberg's theorem alone is not sufficient).

8.3.3 Theorem (Colored Tverberg theorem). *For any integers $r, d \geq 2$ there exists an integer t such that given any $t(d+1)$-point set $Y \subset \mathbf{R}^d$ partitioned into $d+1$ color classes $Y_1, \ldots, Y_{d+1}$ with t points each, there exist r pairwise disjoint sets $A_1, \ldots, A_r$ such that each A_i contains exactly one point of each Y_j, $j = 1, 2, \ldots, d+1$ (that is, the A_i are rainbow), and $\bigcap_{i=1}^r \operatorname{conv}(A_i) \neq \emptyset$.*

Let $T_{\mathrm{col}}(d, r)$ denote the smallest t for which the conclusion of the theorem holds. It is known that $T_{\mathrm{col}}(2, r) = r$ for all r. It is possible that $T_{\mathrm{col}}(d, r) = r$ for all d and r, but only weaker bounds have been proved. The strongest known result guarantees that $T_{\mathrm{col}}(d, r) \leq 2r-1$ whenever r is a prime power.

Recall that in Tverberg's theorem, if we need only the existence of $T(d, r)$, rather than the precise value, several simple arguments are available. In contrast, for the colored version, even if we want only the existence of $T_{\mathrm{col}}(d, r)$, there is essentially only one type of proof, which is not easy and which uses topological methods. Since such methods are not considered in this book, we have to omit a proof of the colored Tverberg theorem.

Bibliography and remarks. Tverberg's theorem was conjectured by Birch and proved by Tverberg (really!) [Tve66]. His original proof is