Spectral Measures, the Spectral Theorem, and Ergodic Theory

Sam Ziegler

The spectral theorem for unitary operators

The presentation given here largely follows [4]. K will refer to the unit circle throughout.

Recall that a measure preserving automorphism (m.p.a.) T on a Borel probability space $(X, \mathcal{B}, \mu)$ gives rise to a unitary map on $L^2(X, \mu)$ via $f \to f \circ T$. Call this map U_T. It is natural to ask if the spectral properties of U_T says something about T as an m.p.a. Before delving into this relationship, let's discuss some facts about unitary operators and the spectral theorem.

Motivation: given a unitary (even normal) operator U on a finite dimensional Hilbert space H, we can find $\lambda_1, \lambda_2, \ldots, \lambda_k$ complex numbers, and mutually orthogonal subspaces $H_1, \ldots, H_k$ such that

$$H = \bigoplus_{i=1}^{k} H_i$$

and

$$U = \bigoplus_{i=1}^{k} \lambda_i P_i$$

where P_i is the projection onto H_i.

Or: if H is n-dimensional, let X be a set of size n with counting measure μ. Then $L^2(X, \mu)$ is just $\mathbb{C}^n$, and fin. dim. spectral theorem says that U acts on $L^2(X, \mu)$ via

$$U(g)(x) = f(x)g(x)$$

where $f(x)$ runs through the eigenvalues of U as x runs through the finite set X. All the spectral theorem for normal operators does is extend this to infinite dimensional H, where X will now be a non-trivial measure space (i.e. $\sigma(U) \subseteq \mathbb{C}$).

We could consider both the strong and weak topology on $U(\mathcal{H})$. In fact, these topologies agree. Consider a sequence (U_n) converging weakly to U. So for any $\xi \in \mathcal{H}$ we have

$$(U_n\xi, U\xi) \to (U\xi, U\xi) = ||U\xi||^2$$

so that

$$(U_n\xi - U\xi, U_n\xi - U\xi) = ||U_n\xi||^2 + ||U\xi||^2 - 2\mathrm{Re}(U_n\xi, U\xi) \to 0$$

using the fact that U_n and U preserve norm. So weak convergence implies strong convergence.

Definition. We say that a sequence of complex numbers $\{r_n\}$ is **positive definite** if for all sequences $\{a_n\}$ and all nonnegative integers N,

$$\sum_{n,m=0}^{N} r_{n-m} a_n \overline{a}_m \geq 0$$

Notice that if U is a unitary operator on a Hilbert space H, and $x \in H$, then the sequence $r_n = (U^n x, x)$ is a positive definite sequence. For

$$\begin{aligned} \sum_{n,m=0}^{N} (U^{n-m}x, x) a_n \overline{a}_m &= \sum_{n,m=0}^{N} (U^n x, U^m x) a_n \overline{a}_m \\ &= (\sum_{i=0}^{N} a_i U^i x, \sum_{i=0}^{N} a_i U^i x) \geq 0 \end{aligned}$$

Theorem (Herglotz). If $\{r_n\}$ is a positive definite sequence then there is a unique finite, non-negative measure μ on the circle (or $[0, 1)$ such that

$$r_n = \int_K z^n d\mu$$

That is, the $\{r_n\}$ are the fourier coefficients of the measure.

(sidenote: when it comes to providing spectral measures for the transformations U_T, there is a more direct way to do it than use Herglotz's theorem.)

The proof uses the following facts: positive definite sequences are bounded, z^n has integral 0 around the unit circle for $n \neq 0$, with respect to Lebesgue measure, and the polynomials $p(z)$ are dense in the space of continuous functions. Along with the Riesz-Markov representation theorem and a couple algebraic maneuvers, this gives us the existence part of this theorem.

Also note, knowing $\int_K z^n d\mu$ for all $n \in \mathbb{Z}$ completely determines μ as a member of $C(K)^*$, since the polynomials are dense in this space.

Theorem (Wiener). Let μ be a finite Borel measure defined on K. If H is a closed subspace of $L^2(K,\mu)$ such that H is invariant under the map $f(z) \mapsto zf(z)$, then H is necessarily of the form

$$H = \chi_B L^2(K,\mu) = \{f \in L^2(K,\mu) : \text{supp } f \subset B\}$$

Let U be a unitary operator on H and let $x \in H$. Then let $Z(x)$ be the closed linear span of the set $\{U^n x : n \in \mathbb{Z}\}$. So $Z(x)$ is what we call a cyclic subspace of H with respect to U, with cyclic-vector x. $Z(x)$ is a reducing subspace for U (noting that $Z(x)$ is invariant under both U and $U^{-1} = U^*$). Also, let μ_x be the measure we get from Hergoltz's theorem.

(Aside: not every invariant subspace for a unitary map is reducing. Consider $U : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ the right shift. Let $M = \{x \in \ell^2(\mathbb{Z}) : x(n) = 0 \text{ for } n \leq 0\}$. Then M is U invariant but not reducing.)

Claim. $U|Z(x)$ is unitarily equivalent to the map $V_x : L^2(K,\mu_x) \to L^2(K,\mu_x)$ defined by $(V_x f)(z) = zf(z)$.

That is, on the cyclic subspace $Z(x)$, U behaves just like a multiplication operator; indeed, multiplication by the identity function.

(sketch). Just define W on the set $\{U^n x\}$ by $W(U^n x) = z^n$. Then by defnition this is inner product preserving. Now extend to linear combinations and take closures to get an isometry onto $L^2(K,\mu_x)$, as the polynomial functions are sup-norm dense in $C(K)$, and furthermore, the continuous functions are dense in $L^2(K,\mu_x)$.

$\square$

As a result of this construction, we get a correspondence between cyclic subspaces and measures on K. There are various properties to prove. For example, $U|Z(x) \approx U|Z(y)$ (unitarily equivalent) iff $\mu_x \approx \mu_y$. By the above it suffices to show $V_x \approx V_y$ iff $\mu_x \approx \mu_y$. But if $WV_x = V_y W$ for some unitary W, let $f(z)$ be the function $W(1)$, where 1 is the constant function of $L^2(K,\mu_x)$. Then $WV_x^n 1 = V_y^n f$, which is to say $W(z^n) = f(z)z^n$. So by density considerations, $W(g) = f \cdot g$ for all $g \in L^2(K,\mu_x)$. But W is an isometry, so given $B \subset K$ measurable we have

$$\mu_x(B) = \int_K |\chi_B|^2 d\mu_x = \int_K |f|^2 |\chi_B|^2 d\mu_y = \int_B |f|^2 d\mu_y$$

therefore $\mu_x \ll \mu_y$. By symmetry $\mu_x \approx \mu_y$.

On the other hand if $\mu_x \approx \mu_y$ then we can define an isometry by $g \in L^2(K,\mu_x) \to g(\frac{d\mu_x}{d\mu_y})^{\frac{1}{2}}$.

There are several other facts one needs to check in order to prove the theorem but I omit them for the sake of brevity.

If U is a unitary operator on a separable Hilbert space H, then there exists a maximal cyclic subspace. Indeed, for any $x \in H$ there is a maximal cyclic subspace containing x.

Claim. If U_i are unitary operators on this Hilbert spaces H_i such that $U_1 \approx U_2$ and $U_1|Z(x) \approx U_2|Z(y)$ then $U_1|Z(x)^\perp \approx U_2|Z(y)^\perp$.

Proof omitted for now.

Finally, let $\{e_n\}$ be an orthonormal basis for H. Let $Z(x_1)$ be a cyclic subspace containing e_1. Let $Z(x_2)$ be a maximal cyclic subspace of $Z(x_1)^\perp$ containing $(I - P_1)e_2$, the projection of e_2 onto $Z(x_1)^\perp$. Note $e_1 \in Z(x_1)$, $e_2 \in Z(x_1) \oplus Z(x_2)$. Repeating this for all n, we get $e_n \in Z(x_1) \oplus Z(x_2) \oplus \ldots \oplus Z(x_n)$ and, since the e_n span the space, we get

$$H = \sum_n \bigoplus_{i<n} Z(x_i)$$

By maximality, we have $\mu_{x_1} \succeq \mu_{x_2} \succeq \ldots$. This is because if $x, y \in H$ and $\mu_x \perp \mu_y$, then $Z(x+y) = Z(x) \oplus Z(y)$, and so if $Z(x)$ is maximal, for any $y \neq 0$, we must have $y \not\perp x$. This means $y \preceq x$ for all y (measures split into absolutely continuous and perpendicular components.)

Now note: since we are only concerned about the identity of these measures up to measure equivalence, the above data can be summarized by providing μ_{x_1} and $A_n = \operatorname{supp}(\frac{d\mu_{x_n}}{d\mu_{x_{n-1}}})$. Note $A_2 \supseteq A_3 \supseteq A_4 \subseteq \ldots$, because μ_{x_1} restricted to A_n is measure equivalent to μ_{x_n}. The function

$$M = \sum_{n=1}^{\infty} \chi_{A_n}$$

where $A_1 = K$, is called the multiplicity function.

At last, this is the classification we are after: a unitary transformation is determined up to unitary equivalence by this infinite list of decreasing measures (also up to measure equivalence).

One can check: measure equivalence is a Borel relation. Note that if σ and ν are finite measures on some (standard Borel) space $(X, \mathcal{B})$, then

$$\sigma \ll \nu \iff (\forall \epsilon > 0)(\exists \delta > 0)(\forall A \in \mathcal{B})\ (\nu(A) < \delta \Rightarrow \sigma(A) < \epsilon)$$

The map taking (U, x) to μ_x is also Borel.

$$\nu = \mu_x \iff (\forall n) \left(\int_K z^n d\nu = (U^n x, x) \right)$$

This is enough to specify the measure because the polynomials are dense in $C(K)$. Kechris says that the following will define the maximal spectral type of U: let (e_n) be an orthonormal basis for H. Then we have

$$\nu \approx \mu_{x_1} \iff \left(\int_K z^n d\nu = \sum_{k=1}^{\infty} \frac{1}{2^k} (U^n e_k, e_k) \right)$$

i.e. $\mu_{x_1} \approx \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_{e_n}$.

I still need to verify that the map $U(H) \to \mathcal{M}^\omega$ is Borel...

0.1 Descriptive results

Note that since the measure equivalence relation on $\mathcal{M}$ is Borel, the relation of unitary equivalence on $U(H)$ is Borel by the spectral theorem. One must check that the above assignment of measures to a unitary operator is Borel. This means that there can be no Borel reduction of the conjugacy relation on ergodic transformations to equivalence of unitary operators (the former being non-Borel by a theorem of Foreman, Rudolph, and Weiss). On the other hand, Kechris has shown that measure equivalence does reduce to ergodic isomorphism. That is

$$(\cong^u) <_B (\cong^{\text{mpt}}).$$

In their paper [3], Kechris and Sofronidis show that the relation of measure equivalence on the space $P(Y)$ for Y Polish is turbulent. This means in particular that the relation of unitary equivalence of unitary transformations is not classifiable by countable structures, i.e. by object in a Borel S_∞ space.

Spectral measures

Definition. Given a measure space $(X, \mathcal{B})$, and a Hilbert space $\mathcal{H}$, a **spectral measure** is a function $P : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ taking projections as values such that

(i) $P(\emptyset) = 0$

(ii) $P(X) = 1_{\mathcal{H}}$

(iii) For every sequence $E_1, E_2, \ldots$ of pairwise disjoint sets, we have

$$P(E_1 \cup E_2 \cup \ldots) = \sum_{n=1}^{\infty} P(E_n)$$

where the convergence is in the sense of the strong topology on $\mathcal{B}(\mathcal{H})$.

Now, given a normal operator $N : \mathcal{H} \to \mathcal{H}$, there is a natural way to define a spectral measure. There is a Borel functional calculus for N; that is there is a representation $\pi : B(\sigma(N)) \to \mathcal{B}(\mathcal{H})$ which extends the usual continuous functional calculus for normal operators (the latter is itself a consequence of the general fact that a commutative C^* algebra is $*-$isomorphic to the continuous functions on its Gelfand spectrum - see [1]). Now, define $P(E) = \chi_E(N)$.

The spectral theorem for unitary operators appearing in [2] generalizes the case for $\mathbb{Z}$. For locally compact group G, the **character group** of G, denoted G^* (or $\widehat{G}$), is the collection of continuous homomorphisms from G into S^1 with the topology of compact convergence (equivalently pointwise convergence?). Note that if G is discrete, the characters are simply homorphisms; but being a homomorphism is a closed condition on $(S^1)^G$ with the product topology (pointwise convergence). Thus G^* is compact (assuming the topology is indeed pointwise convergence).

Note: if the Fourier coefficients of a sequence of measures on S^1 converges (pointwise on $\mathbb{Z}$), then the measures converge weakly to some measure on the circle (see Rudin on Katznelson, Helson, etc...).

References

[1] W. Arveson, *A short course on spectral theory.* Springer-Verlag, New York, 2002

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[4] W. Parry *Topics in ergodic theory* Cambridge University Press, Cambridge, UK, 1981