Spectral Measures, the Spectral Theorem, and Ergodic Theory

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The spectral theorem for unitary operators

The presentation given here largely follows [4]. $K$ will refer to the unit circle throughout.

Recall that a measure preserving automorphism (m.p.a.) $T$ on a Borel probability space $(X,\mathcal{B},\mu)$ gives rise to a unitary map on $L^2(X,\mu)$ via $f \to f \circ T$. Call this map $U_T$. It is natural to ask if the spectral properties of $U_T$ says something about $T$ as an m.p.a. Before delving into this relationship, let’s discuss some facts about unitary operators and the spectral theorem.

Motivation: given a unitary (even normal) operator $U$ on a finite dimensional Hilbert space $H$, we can find $\lambda_1, \lambda_2, \ldots, \lambda_k$ complex numbers, and mutually orthogonal subspaces $H_1, \ldots, H_k$ such that

$$H = \bigoplus_{i=1}^{k} H_i$$

and

$$U = \bigoplus_{i=1}^{k} \lambda_i P_i$$

where $P_i$ is the projection onto $H_i$.

Or: if $H$ is n-dimensional, let $X$ be a set of size $n$ with counting measure $\mu$. Then $L^2(X,\mu)$ is just $\mathbb{C}^n$, and fin. dim. spectral theorem says that $U$ acts on $L^2(X,\mu)$ via

$$U(g)(x) = f(x)g(x)$$

where $f(x)$ runs through the eigenvalues of $U$ as $x$ runs through the finite set $X$. All the spectral theorem for normal operators does is extend this to infinite dimensional $H$, where $X$ will now be a non-trivial measure space (i.e. $\sigma(U) \subseteq \mathbb{C}$).

We could consider both the strong and weak topology on $U(\mathcal{H})$. In fact, these topologies agree. Consider a sequence $(U_n)$ converging weakly to $U$. So for any $\xi \in \mathcal{H}$ we have

$$(U_n \xi, U \xi) \to (U \xi, U \xi) = ||U \xi||^2$$
so that

\[(U_n \xi - U \xi, U_n \xi - U \xi) = ||U_n \xi||^2 + ||U \xi||^2 - 2 \text{Re}(U_n \xi, U \xi) \to 0\]

using the fact that \(U_n\) and \(U\) preserve norm. So weak convergence implies strong convergence.

**Definition.** We say that a sequence of complex numbers \(\{r_n\}\) is **positive definite** if for all sequences \(\{a_n\}\) and all nonnegative integers \(N\),

\[
\sum_{n,m=0}^{N} r_{n-m} a_n \overline{a_m} \geq 0
\]

Notice that if \(U\) is a unitary operator on a Hilbert space \(H\), and \(x \in H\), then the sequence \(r_n = (U^n x, x)\) is a positive definite sequence. For

\[
\sum_{n,m=0}^{N} (U^{n-m} x, x)a_n \overline{a_m} = \sum_{n,m=0}^{N} (U^n x, U^m x)a_n \overline{a_m}
\]

\[
= (\sum_{i=0}^{N} a_i U^i x, \sum_{i=0}^{N} a_i U^i x) \geq 0
\]

**Theorem (Herglotz).** If \(\{r_n\}\) is a positive definite sequence then there is a unique finite, non-negative measure \(\mu\) on the circle (or \([0,1]\)) such that

\[r_n = \int_{K} z^n d\mu\]

That is, the \(\{r_n\}\) are the fourier coefficients of the measure.

(sidenote: when it comes to providing spectral measures for the transformations \(U_T\), there is a more direct way to do it than use Herglotz’s theorem.)

The proof uses the following facts: positive definite sequences are bounded, \(z^n\) has integral 0 around the unit circle for \(n \neq 0\), with respect to Lebesgue measure, and the polynomials \(p(z)\) are dense in the space of continuous functions. Along with the Riesz-Markov representation theorem and a couple algebraic maneuvers, this gives us the existence part of this theorem.

Also note, knowing \(\int_{K} z^n d\mu\) for all \(n \in \mathbb{Z}\) completely determines \(\mu\) as a member of \(C(K)^*\), since the polynomials are dense in this space.
Theorem (Wiener). Let $\mu$ be a finite Borel measure defined on $K$. If $H$ is a closed subspace of $L^2(K, \mu)$ such that $H$ is invariant under the map $f(z) \mapsto zf(z)$, then $H$ is necessarily of the form

$$H = \chi_B L^2(K, \mu) = \{ f \in L^2(K, \mu) : \text{supp } f \subset B \}$$

Let $U$ be a unitary operator on $H$ and let $x \in H$. Then let $Z(x)$ be the closed linear span of the set $\{U^n x : n \in \mathbb{Z}\}$. So $Z(x)$ is what we call a cyclic subspace of $H$ with respect to $U$, with cyclic-vector $x$. $Z(x)$ is a reducing subspace for $U$ (noting that $Z(x)$ is invariant under both $U$ and $U^{-1} = U^*$). Also, let $\mu_x$ be the measure we get from Hergoltz’s theorem.

(Aside: not every invariant subspace for a unitary map is reducing. Consider $U : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ the right shift. Let $M = \{ x \in \ell^2(\mathbb{Z}) : x(n) = 0 \text{ for } n \leq 0 \}$. Then $M$ is $U$ invariant but not reducing.)

Claim. $U|Z(x)$ is unitarily equivalent to the map $V_x : L^2(K, \mu_x) \to L^2(K, \mu_x)$ defined by $(V_x f)(z) = zf(z)$.

That is, on the cyclic subspace $Z(x)$, $U$ behaves just like a multiplication operator; indeed, multiplication by the identity function.

(sketch). Just define $W$ on the set $\{U^n x\}$ by $W(U^n x) = z^n$. Then by definition this is inner product preserving. Now extend to linear combinations and take closures to get an isometry onto $L^2(K, \mu_x)$, as the polynomial functions are sup-norm dense in $C(K)$, and furthermore, the continuous functions are dense in $L^2(K, \mu_x)$.

As a result of this construction, we get a correspondence between cyclic subspaces and measures on $K$. There are various properties to prove. For example, $U|Z(x) \approx U|Z(y)$ (unitarily equivalent) iff $\mu_x \approx \mu_y$. By the above it suffices to show $V_x \approx V_y$ iff $\mu_x \approx \mu_y$. But if $WV_x = V_y W$ for some unitary $W$, let $f(z)$ be the function $W(1)$, where $1$ is the constant function of $L^2(K, \mu_x)$. Then $WV_x^n 1 = V_y^n f$, which is to say $W(z^n) = f(z)z^n$. So by density considerations, $W(g) = f \cdot g$ for all $g \in L^2(K, \mu_x)$. But $W$ is an isometry, so given $B \subset K$ measurable we have

$$\mu_x(B) = \int_K |\chi_B|^2 d\mu_x = \int_K |f|^2 |\chi_B|^2 d\mu_y = \int_B |f|^2 d\mu_y$$

therefore $\mu_x \ll \mu_y$. By symmetry $\mu_x \approx \mu_y$.

On the other hand if $\mu_x \approx \mu_y$ then we can define an isometry by $g \in L^2(K, \mu_x) \to g(\frac{d\mu_x}{d\mu_y})^{1/2}$. 

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There are several other facts one needs to check in order to prove the theorem but I omit them for the sake of brevity.

If $U$ is a unitary operator on a separable Hilbert space $H$, then there exists a maximal cyclic subspace. Indeed, for any $x \in H$ there is a maximal cyclic subspace containing $x$.

**Claim.** If $U_i$ are unitary operators on this Hilbert spaces $H_i$ such that $U_1 \approx U_2$ and $U_1|Z(x) \approx U_2|Z(y)$ then $U_1|Z(x)^\perp \approx U_2|Z(y)^\perp$.

Proof omitted for now.

Finally, let $\{e_n\}$ be an orthonormal basis for $H$. Let $Z(x_1)$ be a cyclic subspace containing $e_1$. Let $Z(x_2)$ be a maximal cyclic subspace of $Z(x_1)^\perp$ containing $(I - P_1)e_2$, the projection of $e_2$ onto $Z(x_1)^\perp$. Note $e_1 \in Z(x_1)$, $e_2 \in Z(x_1) \oplus Z(x_2)$. Repeating this for all $n$, we get $e_n \in Z(x_1) \oplus Z(x_2) \oplus \ldots \oplus Z(x_n)$ and, since the $e_n$ span the space, we get

$$H = \sum_n \bigoplus Z(x_i)$$

By maximality, we have $\mu_{x_1} \succeq \mu_{x_2} \succeq \ldots$. This is because if $x, y \in H$ and $\mu_x \perp \mu_y$, then $Z(x + y) = Z(x) \oplus Z(y)$, and so if $Z(x)$ is maximal, for any $y \neq 0$, we must have $y \not\perp x$. This means $y \preceq x$ for all $y$ (measures split into absolutely continuous and perpendicular components.)

Now note: since we are only concerned about the identity of these measures up to measure equivalence, the above data can be summarized by providing $\mu_{x_1}$ and $A_n = \text{supp}(\frac{d\mu_{x_n}}{d\mu_{x_{n-1}}})$. Note $A_2 \supseteq A_3 \supseteq A_4 \subseteq \ldots$, because $\mu_{x_1}$ restricted to $A_n$ is measure equivalent to $\mu_{x_n}$. The function

$$M = \sum_{n=1}^{\infty} \chi A_n$$

where $A_1 = K$, is called the multiplicity function.

At last, this is the classification we are after: a unitary transformation is determined up to unitary equivalence by this infinite list of decreasing measures (also up to measure equivalence).

One can check: measure equivalence is a Borel relation. Note that if $\sigma$ and $\nu$ are finite measures on some (standard Borel) space $(X, \mathcal{B})$, then

$$\sigma \ll \nu \iff (\forall \epsilon > 0)(\exists \delta > 0)(\forall A \in \mathcal{B}) (\nu(A) < \delta \Rightarrow \sigma(A) < \epsilon)$$

The map taking $(U, x)$ to $\mu_x$ is also Borel.

$$\nu = \mu_x \iff (\forall n) \left( \int_K z^n d\nu = (U^n x, x) \right)$$
This is enough to specify the measure because the polynomials are dense in $C(K)$. Kechris says that the following will define the maximal spectral type of $U$: let $(e_n)$ be an orthonormal basis for $H$. Then we have

$$\nu \approx \mu_{x_1} \iff \left( \int_K z^n\,d\nu = \sum_{k=1}^{\infty} \frac{1}{2^k} (U^ne_k, e_k) \right)$$

i.e. $\mu_{x_1} \approx \sum_{n=1}^{\infty} \frac{1}{2^n} \mu_{e_n}$.

I still need to verify that the map $U(H) \to \mathcal{M}^\omega$ is Borel.

### 0.1 Descriptive results

Note that since the measure equivalence relation on $\mathcal{M}$ is Borel, the relation of unitary equivalence on $U(H)$ is Borel by the spectral theorem. One must check that the above assignment of measures to a unitary operator is Borel. This means that there can be no Borel reduction of the conjugacy relation on ergodic transformations to equivalence of unitary operators (the former being non-Borel by a theorem of Foreman, Rudolph, and Weiss). On the other hand, Kechris has shown that measure equivalence does reduce to ergodic isomorphism. That is

$$(\cong^u) <_B (\cong^\text{mpt}).$$

In their paper [3], Kechris and Sofronidis show that the relation of measure equivalence on the space $P(Y)$ for $Y$ Polish is turbulent. This means in particular that the relation of unitary equivalence of unitary transformations is not classifiable by countable structures, i.e. by object in a Borel $S^\infty$ space.

### Spectral measures

**Definition.** Given a measure space $(X, \mathcal{B})$, and a Hilbert space $\mathcal{H}$, a **spectral measure** is a function $P : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ taking projections as values such that

(i) $P(\emptyset) = 0$

(ii) $P(X) = 1_{\mathcal{H}}$

(iii) For every sequence $E_1, E_2, \ldots$ of pairwise disjoint sets, we have

$$P(E_1 \cup E_2 \cup \ldots) = \sum_{n=1}^{\infty} P(E_n)$$

where the convergence is in the sense of the strong topology on $\mathcal{B}(\mathcal{H})$. 

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Now, given a normal operator $N : \mathcal{H} \to \mathcal{H}$, there is a natural way to define a spectral measure. There is a Borel functional calculus for $N$; that is there is a representation $\pi : \mathcal{B}(\sigma(N)) \to \mathcal{B}(\mathcal{H})$ which extends the usual continuous functional calculus for normal operators (the latter is itself a consequence of the general fact that a commutative $C^*$ algebra is $*$-isomorphic to the continuous functions on its Gelfand spectrum - see [1]). Now, define $P(E) = \chi_E(N)$.

The spectral theorem for unitary operators appearing in [2] generalizes the case for $\mathbb{Z}$. For locally compact group $G$, the **character group** of $G$, denoted $G^*$ (or $\widehat{G}$), is the collection of continuous homomorphisms from $G$ into $S^1$ with the topology of compact convergence (equivalently pointwise convergence?). Note that if $G$ is discrete, the characters are simply homomorphisms; but being a homomorphism is a closed condition on $(S^1)^G$ with the product topology (pointwise convergence). Thus $G^*$ is compact (assuming the topology is indeed pointwise convergence).

Note: if the Fourier coefficients of a sequence of measures on $S^1$ converges (pointwise on $\mathbb{Z}$), then the measures converge weakly to some measure on the circle (see Rudin on Katznelson, Helson, etc…) .

References


