Math 586 - Computational Finance

Solving the Pricing Equation for an Annuity

A related problem, called an annuity, is to find the present value $P(0)$ of a series of N yearly fixed payments of c (on the last day of the year). Here r is the interest rate and the compounding is continuous. The pricing problem is a FVP of the form

$$\frac{dP}{dt} = rP - c\sum_{j=1}^{N-1} \delta(t-j), \quad t < N, \text{ with } P(N) = c. \tag{1}$$

We solve (1) by first re-arranging the terms and then multiplying by an integrating factor e^{-rt} to obtain

$$e^{-rt}\{\frac{dP}{dt} - rP\} = -ce^{-rt}\sum_{j=1}^{N-1} \delta(t-j).$$

We recognize that the left side of the equation is the derivative of a product so that we have

$$\frac{d}{dt}[e^{-rt}P] = -ce^{-rt}\sum_{j=1}^{N-1} \delta(t-j)$$

We are now able to integrate both sides of the equation from 0 to N with respect to t to obtain

$$e^{-rN}P(N) - P(0) = -c\int_0^N e^{-rt}\sum_{j=1}^{N-1} \delta(t-j)dt.$$

Using the properties of the δ function, the final condition $P(N) = c$, and solving for $P(0)$ we obtain

$$P(0) = c\sum_{j=1}^{N-1} e^{-rj} + ce^{-rN} = c\sum_{j=1}^{N} e^{-rj}.$$

This is the standard formula for the present value of this annuity.