

## Financial Mathematics - Spring 2006 Homework - 4

1. Consider the following random walk in discrete time. The state variable  $S_j$  takes values in  $\{\dots, -2, -1, 0, 1, 2, \dots\}$ , and satisfies the stochastic difference equation

$$S_{i+1} = S_i + R_{i+1}, \quad S_0 = 0,$$

where the  $R_i$  are i.i.d. random variables such that

$$R_j = \begin{cases} 1 & \text{w.p. } r \\ -1 & \text{w.p. } r \\ 0 & \text{w.p. } 1 - 2r \end{cases}$$

- (a) Derive the forward Kolmogorov equation for the conditional probability  $p_n(m) = \Pr[S_n = m | S_0 = 0]$ .  
 (b) Take the continuous limit of the forward equation and derive the forward diffusion equation for the density function  $u$ . Use the scaling to continuous variables  $t$  and  $x$  that we used in class. State the complete diffusion equation problem for the  $u$ .
2. Consider the SDE:

$$dS = \mu dt + \sigma dW, \quad S(0) = S_0,$$

where  $W(t)$  is Brownian motion.

- (a) Find a strong solution for  $S(t)$ .  
 (b) Find the density function  $p(S, t; S_0, 0)$ .  
 (c) Using the strong solution in (a), introduce discrete time steps with step size  $\Delta t$ . In addition, replace  $dW$  by an appropriate normal random variable  $\phi$ . This equation can be used to generate simulations of the process  $S(t)$ .  
 (d) Modify Higham's Matlab code on the next page to generate 10 sample paths of the discretized process in the same figure. Assume  $S_0 = 2$ ,  $\mu = 0.3$ , and  $\sigma = 0.1$ . Your simulation should start at  $t = 0$  and stop at  $T = 1$  using 100 steps. In the same figure, please include a graph of the mean.
3. If  $dS = \mu S dt + \sigma S dW$ , find the SDE for  $Y$ , i.e.  $dY$ , if:

$$(a) \ Y = AS, \ A \text{ constant}, \quad (b) \ Y = S^n.$$

4. Let  $Y(t)$  be a random process with probability density function

$$p(y, t) = \frac{e^{-(\ln y + t/2)^2 / 2t}}{\sqrt{2\pi t y}}, \quad y \in (0, \infty)$$

Find  $E[Y]$  and  $\text{Var}(Y)$ .

5. We showed that the risk-neutral price of a lognormal asset satisfied the SDE:  $dS = rS dt + \sigma S dW$ , where  $r$  is the risk-free rate and  $\sigma$  is the volatility. Show that  $E^Q[S_T] =$  forward price at time  $t$  for delivery date  $T$ . Here the expectation is with respect to the risk-neutral density, which you must find from the SDE.

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%Higham Program --- Chapter 7
%
% Plot discrete sample paths

randn('state',sum(100*clock))
clf

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Problem parameters %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
S = 1; mu = 0.05; sigma = 0.5; L = 1e2; T = 1; dt = T/L; M = 50;
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tvals = [0:dt:T];
Svals = S*cumprod(exp((mu-0.5*sigma^2)*dt + sigma*sqrt(dt)*randn(M,L)),2);
Svals = [S*ones(M,1) Svals]; % add initial asset price
plot(tvals,Svals)
title('50 asset paths')
xlabel('t'), ylabel('S(t)')

```