Nonlinear Equations

1. State Newton’s Method iteration for find the root of \( f(x) = 4x^3 - 1 - \exp(x^2/2) = 0 \) near \( x = 1 \).

2. Explain how you would find the roots of \( f(x) = x^2 + 2x + 5 \).

3. Recast Newton’s method in terms of a function \( g(x) \) so that fixed points of \( g(x) \) are the roots of \( f(x) \).

4. Assuming that \( f(0) = 0 \) and \( f'(0) \neq 0 \) show that Newton’s method if started sufficiently close to \( x = 0 \) will converge at least with a quadratic rate to 0. Use the formulation in the previous part.

5. Why is the secant method more efficient (less expensive) than Newton’s method?

Interpolation

1. Why is it a bad idea to use high degree (\( \geq 6 \) say) interpolating polynomials?

2. Generate a divided difference table for the data: \((-1, 1), (0, 0), (3, 9), (5, 25)\).

3. If the entries in a divided difference table are zero beyond a certain point, what does that tell you?

4. How is the \( n^{th} \) order Newton divided difference \( f[x_0, x_1, \ldots, x_n] \) related to some derivative of \( f(x) \)?

5. Find an interpolating polynomial of the lowest degree that goes through the points \((0, 0), (1, 1), (2, 2), (3, 3), (4, 4)\). Explain why your answer must be correct.

6. Find the natural cubic spline that goes through the points \((0, 1), (1, 2), (2, 3), (3, 4), (4, 5)\). Explain why your answer must be correct.

7. Why is the Newton divided difference form preferred over the Lagrange basis for the computation of the interpolating polynomials.

Linear Systems

1. Explain why you would always want to use Gaussian elimination (or some form of LU decomposition) instead of Cramer’s rule to solve a dense system of linear equations.

2. If the relative error in the residuals of the solution of a linear system is small what can you say about the relative error of the solution? Explain using a formula.
3. For what type of problems it is advantageous to use an iterative method over Gaussian elimination.

4. What is **diagonal dominance** and how does it relate to iterative methods.

5. Consider

\[ A = \begin{bmatrix} -89 & -2 \\ 230 & 10 \end{bmatrix} \]

Use the shifted inverse power method to compute an approximation to the eigenvalue of \( A \) closest to \(-80\) and its corresponding eigenvector. Be sure to use the LU form for iteration.

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</thead>
<tbody>
<tr>
<td>eigen vector</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>eigval of shift inv.</td>
<td>-</td>
<td></td>
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</tbody>
</table>

Eigenvalue of \( A \) closest to \(-80\): _____________, Eigenvector: [ ]

**Differentiation and Integration**

1. Apply Richardson extrapolation to compute a second-order approximation for the derivative of \( f(x) \). Use forward differences in your first-order approximation and at step size \( h \), Evaluate all the formulas above numerically to find the derivative of \( f(x) = \arctan(x) \) at \( x = 0.3 \) using \( h = 0.1 \). Show the intermediate results with six decimal places. Continue the calculations with the intermediate rounded results.

2. Determine the values of \( a \) and \( b \) in the two point **Gauss rule** (you must derive the result!)

\[ \int_{-1}^{1} f(x) dx \simeq f(a) + f(b) \]

3. Compute the value of

\[ \int_{0}^{0.3} \exp(-x^2) \, dx \]

(a) By the trapezoidal rule using 3 panels on the interval of integration:

(b) By the trapezoidal rule using 6 panels (Try to minimize the number of new functional evaluations necessary to compute the result by using the result from part a).

(c) By the **Rhomberg** improvement method using the results in a) and b).

(d) Evaluate the above integral using the two point **Gauss rule**.
Differential Equations

1. Demonstrate Euler’s method graphically. Be able to do a computational example using Euler’s and Modified Euler’s methods. Explain (or demonstrate graphically) when Euler’s method might provide very large errors.

2. Using the Modified Euler Method, numerically approximate the solution to the nonlinear IVP by

\[ y'(x) = -x^2 \cdot y^2, \text{ with } y(0.75) = 2.3, \]

with \( h = 0.1 \) up to the approximation for \( y(0.95) \). Tabulate

\[ \{n, x, y, f(x_n, y_n), f(x_{n+1}, y_{n+1})\} \]

for \( x = 0.75 \) to 0.85. Use 4 digit exam precision: round recorded intermediate results only to 4 significant decimal digits (5 or higher gets rounded up, and continue calculations with these results.)