Nonlinear Equations

1. State Newton's Method iteration for find the root of $f(x) = 4x^3 - 1 - \exp(x^2/2) = 0$ near $x = 1$.

2. Explain how you would find the roots of $f(x) = x^2 + 2x + 5$.

3. Recast Newton's method in terms of a function $g(x)$ so that fixed points of $g(x)$ are the roots of $f(x)$.

4. Assuming that $f(0) = 0$ and $f'(0) \neq 0$ show that Newton's method if started sufficiently close to $x = 0$ will converge at least with a quadratic rate to 0. Use the formulation in the previous part.

5. Why is the **secant method** more efficient (less expensive) than **Newton's method**?

Interplolation

1. Why is it a bad idea to use high degree (≥ 6 say) interpolating polynomials?

2. Generate a divided difference table for the data: $(-1, 1), (0, 0), (3, 9), (5, 25)$.

3. If the entries in a divided difference table are zero beyond a certain point, what does that tell you?

4. How is the n^{th} order Newton divided difference $f[x_0, x_1, \ldots, x_n]$ related to some derivative of $f(x)$?

5. Find an interpolating polynomial of the lowest degree that goes through the points $(0, 0), (1, 1), (2, 2), (3, 3), (4, 4)$. Explain why your answer must be correct.

6. Find the **natural cubic spline** that goes through the points $(0, 1), (1, 2), (2, 3), (3, 4), (4, 5)$. Explain why your answer must be correct.

7. Why is the **Newton divided difference form** preferred over the **Lagrange basis** for the computation of the interpolating polynomials.

Linear Systems

1. Explain why you would always want to use **Gaussian eimination** (or some form of **LU** decomposition) instead of **Cramer's rule** to solve a dense system of linear equations.

2. If the relative error in the residuals of the solution of a linear system is small what can you say about the relative error of the solution? Explain using a formula.

3. For what type of problems it is advantageous to use an iterative method over Gaussian elimination.

4. What is **diagonal dominance** and how does it relate to iterative methods.

5. Consider

$$A = \begin{bmatrix} -89 & -2 \\ 230 & 10 \end{bmatrix}$$

Use the shifted inverse power method to compute an approximation to the eigenvalue of A closest to -80 and its corresponding eigenvector. Be sure to use the LU form for iteration.

k	0	1	2	3	4
eigen	1				
vector	1				
eigval of shift inv.	-				

Eigenvalue of A closest to -80: ____________________, Eigenvector: $\begin{bmatrix} \quad \end{bmatrix}$.

Differentiation and Integration

1. Apply Richardson extrapolation to compute a second-order approximation for the derivative of $f(x)$. Use forward differences in your first-order approximation and at step size h, Evaluate all the formulas above numerically to find the derivative of $f(x) = \arctan(x)$ at $x = 0.3$ using $h = 0.1$. Show the intermediate results with six decimal places. Continue the calculations with the intermediate rounded results.

2. Determine the values of a and b in the two point **Gauss rule** (you must derive the result!)

$$\int_{-1}^{1} f(x)dx \simeq f(a) + f(b)$$

3. Compute the value of

$$\int_{0}^{0.3} \exp(-x^2)\, dx$$

   (a) By the trapezoidal rule using 3 panels on the interval of integration:

   (b) By the trapezoidal rule using 6 panels (Try to minimize the number of new functional evaluations necessary to compute the result by using the result from part a).

   (c) By the **Rhomberg** improvement method using the results in a) and b).

   (d) Evaluate the above integral using the two point **Gauss rule**.

Differential Equations

1. Demonstrate Euler's method graphically. Be able to do a computational example using Euler's and Modified Euler's methods. Explain (or demonstrate graphically) when Euler's method might provide very large errors.

2. Using the *Modified Euler Method*, numerically approximate the solution to the nonlinear IVP by

$$y'(x) = -x^2 \cdot y^2, \quad \text{with} \quad y(0.75) = 2.3,$$

   with $h = 0.1$ up to the approximation for $y(0.95)$. Tabulate

$$\{\text{n}, \ \text{x}, \ \text{y}, \ , \quad f(x_n, y_n) \ , \ f(x_{n+1}, y_{n+1})\}$$

   for $x = 0.75$ to 0.85. Use 4 digit exam precision: round recorded intermediate results only to 4 significant decimal digits (5 or higher gets rounded up, and continue calculations with these results.