Physicists Try to Find Order in Chaos

Physical systems governed by nonlinear equations of motion exhibit turbulent or chaotic behavior, but the transitions to chaos seem to be orderly

Many physical systems exhibit a range of dynamical behaviors from smooth and ordered to turbulent and disordered. The transition from laminar fluid flow to turbulence as the velocity of the fluid increases is a notable example that has never succumbed to a comprehensive mathematical analysis. Recently, physicists have turned their attention to certain quite simple nonlinear equations, well known to mathematicians, that also show transitions from order to a special kind of disorder called chaos as a parameter in the equations changes.

Building upon the pioneering work of Mitchell Feigenbaum of Cornell University, theoretical physicists are showing that there is a universal character to the transition to chaos—that is, there are a few numbers that describe the transition, and these numbers are the same for all equations that exhibit the transition. The hope, as yet unproven, is that there may be only a few general classes of transitions. For their part, experimental physicists are coming up with persuasive evidence that real physical systems ranging from simple electrical circuits to complex fluids undergo transitions to chaos in quantitative agreement with the predictions of the theorists—that is, the universal numbers are the same for physical systems of widely varying character.

The mathematics of chaos, says Jerry Gollub of Haverford College and the University of Pennsylvania, provides a new handle on one of the oldest problems in physics: how to understand systems comprising large numbers of particles that exhibit apparently random behavior. In the 1950's and 1960's, interest in fluctuations (random variations in the value of some property of the system) revived as physicists began making detailed experimental and theoretical investigations of critical phenomena (second-order phase transitions such as the disappearance of ferromagnetism in permanent magnets above their Curie temperatures, or of any distinction between gas and liquid above the critical temperatures of fluids).

In the early 1970's, a mathematical technique called the renormalization group (for the development of which Kenneth Wilson of Cornell received this year's Nobel physics prize) provided an accurate way to calculate the properties of physical systems near a critical point. In chaos, the interest is on the time evolution of the states of physical systems. Nonetheless, the change from a smooth, ordered dynamics to a turbulent, disordered one is a kind of phase transition. In retrospect, therefore, it is not surprising that techniques such as the renormalization group can be tailored to the study of chaos, although Feigenbaum encountered some difficulty in getting his early work published.

For the moment, such practical applications as analyzing turbulent fluids in realistic conditions by this method are distant at best, and the emphasis is on studying chaos as a phenomenon in its own right, cautions Eric Siggia, also of Cornell. Nonetheless, it is hard to keep fluids out of mind. The recent investigations of chaos directly relate to an old fluid dynamics conjecture, for example.

In 1971, David Ruelle of the Institute of Advanced Scientific Studies at Bures-sur-Yvette near Paris and Floris Takens of the Mathematics Institute of the State University of Groningen proposed that the onset of turbulence in fluids could be described by a succession of three transitions at most. For laminar flow, the velocity at a point in the fluid would be constant. Then there would be a transition to a state in which the velocity oscillated between two values with a specific frequency. Subsequent transitions would add new frequencies. But they argued that after the appearance of three frequencies, the fluid would become unstable and aperiodic—that is, a Fourier analysis of the time evolution of the velocity would reveal no discrete frequencies. This scenario was a radical departure from the older picture of the late Soviet physicist Lev Landau, who had proposed that an infinite number of new frequencies could be added and that the fluid would never be truly aperiodic. Experiments in 1975 by Gollub and Harry Swinney of the University of Texas at Austin and subsequently by many other investigators corroborate the Ruelle-Takens proposal. No one has been able to see more than three frequency components in any fluid before it became turbulent.

Ruelle and Takens did not describe any particular mechanism for the transition to chaos that could be compared in detail with experiment. The renormalization group permits physicists to construct detailed models. The most recent theoretical work by David Rand, Stellan Ostlund, James Sethna, and Siggia (who were members of a 15-month study of nonequilibrium phenomena by an interdisciplinary group of mathematicians and physicists at the University of California at Santa Barbara's Institute for Theoretical Physics) and by Scott Shenker and Leo Kadanoff of the University of Chicago and Feigenbaum worked out the case that physicists call quasiperiodicity in which chaos ensues after the appearance of two frequencies.

Physicists describe the time evolution of a physical system by a trajectory in a mathematical space whose coordinates are the variables of interest. For fluid flow, the variables could be the components of the velocity for each fluid particle, a rather large number. After a long enough time, the system will settle down to its steady-state orbit (also called an attractor because it attracts all trajectories starting from some range of initial conditions). Chaos involves a special kind of attractor for which Ruelle and Takens coined the term strange attractor. Motion on the strange attractor is unpredictable, although the nonlinear equations of motion are completely deterministic. This unpredictability led James Yorke of the University of Maryland at College Park to apply the word chaos to this behavior several years ago.

A key feature of strange attractors was identified as far back as 1963 by Edward Lorenz of the Massachusetts Institute of Technology, who was interested in long-range weather forecasting. The feature is extreme sensitivity to initial conditions. The idea is that trajectories beginning from arbitrarily close initial conditions will exponentially diverge. The trajectories are all on the strange attractor, but over the course of time they follow quite different paths on it.

Although fluids are naturally described by sets of differential equations, trajectories can be rigorously determined by difference equations or maps that give the value of a variable at one time in terms of its value and the values of the other variables at the preceding time. To obtain a trajectory, one iteratively applies the map to a set of initial values of the variables. In the case of a single variable, the map is one-dimensional and of the form $x_{n+1} = f(x_n, \lambda)$, where λ is a

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parameter that measures the departure of the system from equilibrium and n is the number of the iteration.

The quasiperiodic route to chaos involves trajectories in a space of two dimensions. At small values of the λ parameter, the steady-state trajectory is a fixed point (that is, $x_{n+1} = x_n$, and so on). As λ increases past a critical value, the trajectory oscillates between two points with a certain frequency. This would correspond to the velocity of a point in a fluid, for example, that oscillated between two values with this frequency. As λ increases further past a second critical value, the trajectory becomes quasiperiodic motion on a circular attractor in such a way that it never returns to its initial value. The time dependence of the fluid velocity would be quite complicated, but a Fourier analysis would reveal two fundamental frequencies that are irrational multiples of one another (that is, the frequencies are incommensurate). There would also be peaks in the Fourier power spectrum for all integer combinations of the fundamental frequencies. The final transition at a third critical value of the λ parameter is to chaos.

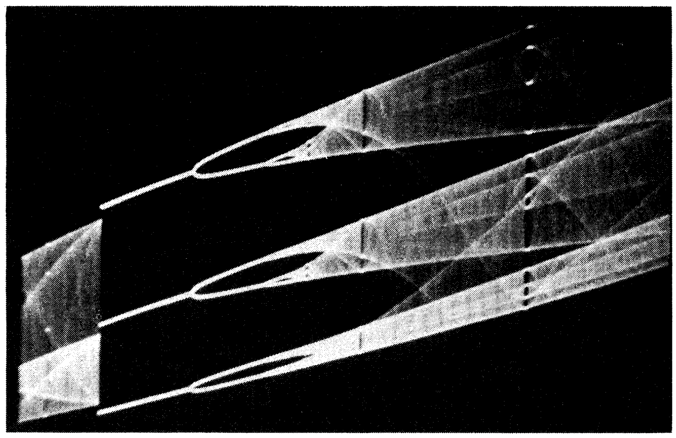
Rand, Ostlund, Sethna, and Siggia and Shenker, Kadanoff, and Feigenbaum considered cases such as those in which the incommensurate frequencies are in the ratio of successive Fibonacci numbers. Using the renormalization group, the two groups independently showed that the transition occurs in a universal way. Here "universal" means that certain numbers that describe the transition to chaos are the same for all maps exhibiting this route to chaos. The theorists have calculated the values of these numbers.

Universality of this type, which is also observed in critical phenomena, is tied in with another property called scale invariance. In the most general case of maps of several dimensions, scale invariance can be seen by looking at the structure of strange attractors. For example, Celso Grebogi, Edward Ott, and Yorke at Maryland have computer studies of several multidimensional maps. One of these is a certain three-dimensional map for which the strange attractor is toroidal. The surface of the toroid is highly wrinkled. Upon taking a closer look, one sees that the pattern of wrinkles continues at all scales of magnification. The situation is completely in the spirit of Jonathan Swift's 1733 jingle "So, naturalists observe, a flea/ Hath smaller fleas that on him prey;/ And these have smaller fleas to bite 'em,/ And so proceed *ad infinitum*," points out Kadanoff.

There are no experimental tests of quasiperiodicity as yet. However, the so-called period doubling route to chaos, which was the object of Feigenbaum's earlier work, has been extensively verified. Feigenbaum analyzed a class of one-dimensional nonlinear maps that exhibit a unique maximum, typified by the logistic map: $x_{n+1} = \lambda\ x_n(1 - x_n)$, where x can occupy any value on the interval 0,1 and λ from 0 to 4. Prior to Feigenbaum's investigations, which were published in 1978 and 1979, it was well known, for example, that the steady-state trajectory is a fixed point for all maps with λ less than a certain value λ_1. But at λ_1, there is a sudden bifurcation to two steady-state solutions. With each iteration of the map, the solution oscillates between the two, such that $x_{n+2} = x_n$. As λ increases further, the two solutions split into four, and so on. At a critical value λ_∞, bifurcation has occurred an infinite number of times, and x randomly assumes any value—that is, chaos arrives. Period doubling has only one fundamental frequency (the others are subharmonics).

Chaos

The current through a nonlinear capacitor in a series LRC circuit follows the period doubling route to chaos. Shown is the period 3 window in which the current oscillates between three values at low voltages, then between six, twelve, and so on until it becomes chaotic as the voltage is raised.



University of California at Berkeley

Feigenbaum analyzed period doubling in one-dimensional maps using empirical methods to uncover the theoretical method, which proved to be a variant of the renormalization group. He showed that period doubling occurs in a universal way; all maps with the same kind of maximum have the same characteristic numbers. Among the characteristic numbers calculated by Feigenbaum are $\delta = 4.6692016\ldots$, which is defined by the values of λ_n at successive bifurcations, and $\alpha = 2.502907875\ldots$, which is defined by the x_n values at successive bifurcations. Subsequently, other characteristic numbers have been calculated by several theorists.

The first really decisive experiment was that in 1980 by Albert Libchaber and Jean Maurer of the Ecole Normale Supérieure in Paris, who investigated Rayleigh-Bénard instabilities in a liquid helium convection cell with a temperature gradient from the bottom to the top. At or near thermal equilibrium, there is no fluid motion, but farther from equilibrium, there is a cyclical flow referred to as a convection roll. Still farther from equilibrium (larger temperature gradients), a secondary instability yields oscillations of the convection roll. One manifestation of the oscillation is a cyclic temperature variation at any one point in the liquid helium.

As the investigators increased the temperature difference (which corresponds to the λ parameter) even more, the period doubling began. The first bifurcation gives rise to a temperature variation with two maxima that are reached on alternating cycles. Subsequent bifurcations give rise to more complex temperature patterns that can be unraveled by Fourier analysis. By this means, Libchaber and Maurer were able to resolve up to four bifurcations. Subsequent bifurcations were not observable because the spacing in the λ parameter becomes too small to control experimentally.

Last year, Marzio Giglio, Sergio Musazzi, and Umberto Perini of the Center of Information, Studies, and Experiments in Milan reported their results for a similar experiment in a water-filled convection cell. They likewise observed four bifurcations, but were also able to estimate the δ parameter and another parameter μ. This number is related to the ratio of the amplitudes of successive subharmonics (Fourier components) of the time-varying temperature as bifurcation proceeds to chaos.

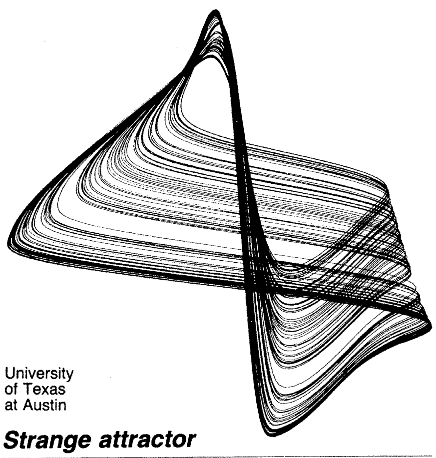
Even more complex period doubling behavior has been quantitatively verified for λ values exceeding λ_∞. Almost a decade ago, Nicholas Metropolis, Myron Stein, and Paul Stein of Los Alamos National Laboratory showed that one-dimensional nonlinear maps of the type analyzed by Feigenbaum exhibited "windows" over certain ranges of λ in which the bifurcation scheme repeated itself. Out of chaos, as it were, a solution of the nonlinear map which repeats every three iterations (period three solution) would emerge, and this would double to six, and so on to a new chaos as the value of λ increased in the window. Similar windows exist in principle for period-doubling sequences of any period. James Testa, José Pérez, and Carson Jeffries of the University of California at Berkeley have recently reported observing several of these windows and the sequences in them.

As compared to a convecting fluid, the physical system studied by these researchers is quite simple. It is a series LRC electrical circuit, which is driven by a sinusoidal voltage source. There are at least three dynamical variables arising from the inductor, the capacitor, and the time-dependent driving voltage. In the Berkeley experiment, the capacitor is a nonlinear device whose capacitance depends on the voltage across it. The amplitude of the sinusoidal applied voltage corresponds to the λ parameter, and the current through the nonlinear capacitor corresponds to the variable x in the nonlinear map. Although simple nonlinear electrical circuits are not themselves of current interest, Gollub comments that investigations of this type do help experimenters to model chaotic behavior.

One of the things Testa, Pérez, and Jeffries could do, for example, is directly measure a bifurcation diagram by plotting the current through the nonlinear capacitor as a function of the amplitude of the driving voltage. The figure shows an oscilloscope trace of the so-called period 3 window. The researchers were also able to make estimates of the values of many of Feigenbaum's universal numbers (δ, α, and μ) in the main (period 1) bifurcation sequence as well as the δ value in the period 3 window.

It is also possible to generate a one-dimensional map experimentally in a system that has many variables. That has been done by Reuben Simoyi, Alan Wolf, and Swinney at Texas. These physicists studied the Belousov-Zhabotinskii reaction, which involves about 25 chemical species. Chemicals flow through a reaction vessel at a fixed rate, which corresponds to the λ parameter. The vessel is well stirred, so the concentrations of the 25 species, which are the variables of interest, are uniform throughout the reactor.

With 25 variables in the system, the resulting trajectory could conceivably be traced, but this is not practical. The Texas researchers drew on an earlier method of Norman Packard, James Crutchfield, Doyne Farmer, and Rob Shaw of the University of California at Santa Cruz, who followed up an idea of Ruelle's that an equivalent trajectory could be obtained by monitoring just one variable in a special way—that is, in a multidimensional plot, where $x(t_k)$, $x(t_k + \tau)$, $x(t_k + 2\tau)$, and so on are the variables. τ is a somewhat arbitary interval. In practice, one limits the number of variables to a number such that adding further variables does not change the form of the trajectory. The figure shows a two-dimensional projection of a three-dimensional strange attractor obtained at Texas in this way by Jean-Claude Roux of the University of Bordeaux I, Simoyi,



Strange attractor

The time evolution of the concentration of bromide ion in the Belousov-Zhabotinskii reaction at t *and* t + *53 seconds yields this dimensional projection of a three-dimensional trajectory.*

and Swinney for the Belousov-Zhabotinskii reaction under investigation. The single variable measured was the concentration of bromide ion.

From the topology of the trajectory, the investigators conjectured that a one-dimensional map would suffice to explain the data, and they were able to deduce its form. Their map possessed a shape similar to that of the logistic map. For example, Simonyi, Wolf, and Swinney observed the same types of windows in the chaotic region that the Berkeley group had seen. Their map successfully reproduced the sequence of x values on a trajectory in a window region that were determined first experimentally and then from the logistic map.

An obvious question is: How does one know whether a physical system will reach chaos by period doubling, by quasiperiodicity, or by some other route? In general, the answer is not known. Jeffries, for example, says he has a drawer full of nonlinear capacitors, and he has no way of knowing which ones will exhibit period doubling other than inserting them into his LRC circuit and trying them out. One recent set of experiments on liquid mercury–filled convection cells by Libchaber and Stephen Fauve of the Ecole Normale Supérieure may provide a clue, however. These researchers added a new feature to the study of the Rayleigh-Bénard instabilities, an applied magnetic field parallel to the axis of the convection rolls. At low field strengths, period doubling predominated, whereas at high fields, a behavior similar to Ruelle-Takens quasiperiodicity prevailed. In between, still other routes to chaos were observed. Libchaber speculates that the effect of the magnetic field on the electrically conducting liquid mercury is to stiffen it against the oscillations. This permits larger temperature gradients to exist before the transition to chaos takes place, which corresponds to a larger λ parameter. In effect, there is a kind of hierarchy of less and less periodic routes to chaos with period doubling occurring in only weakly nonequilibrium systems, and so on.

Several other routes to chaos have been proposed for which some experimental evidence exists. One is called intermittency in which there are alternating periods of stable and chaotic behavior, with no period-doubling cascades at any point. Intermittency was proposed in 1979 by Yves Pomeau and Paul Mannville of the Saclay Nuclear Studies Center near Paris. A detailed analysis resulting in the calculation of characteristic numbers for intermittency has been given by Jorge Hirsch and Douglas Scalapino of Santa Barbara and Bernardo Huberman of the Xerox Palo Alto Research Center. Several others have since done renormalization group calculations.

So, chaos is drawing considerable attention at the moment. Its long-range influence on practical fluid dynamics problems is still speculative, however. Chaos as now understood occurs only in fluid systems whose behavior is effectively low-dimensional by virtue of their confining physical configurations, points out Libchaber. The 1978 experiments in liquid helium convection cells of Günter Ahlers and Robert Behringer (then at Bell Laboratories) clearly show the disappearance of the temporal periodicity characteristic of period doubling and quasiperiodicity prior to the onset of chaos as the width of the cell is increased, for example.

—ARTHUR L. ROBINSON

Additional Reading

Proceedings of the Conference on Order in Chaos, Los Alamos National Laboratory, 24 to 28 May 1982, to be published in *Physica D*.