Math 481 - Applied PDE - Spring 2004
C. Tier

The following topics from *Calculus* and *Linear Algebra* are important and are worth reviewing.

Gradient and Direction Derivative. The *gradient* of a scalar function $z = f(x,y)$ is defined as the vector

$$\nabla f = f_x \mathbf{i} + f_y \mathbf{j}. \tag{1}$$

It is analogous to the derivative of a function in one variable. The *direction derivative* describes the rate of a change of a function in a specific direction. If $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j}$ is a unit vector then the rate of change of f in the direction $\mathbf{n}$ is

$$\frac{df}{d\mathbf{n}} \equiv \nabla f \cdot \mathbf{n} = f_x n_1 + f_y n_2. \tag{2}$$

For example, the rate of change of f in the x-direction is $\nabla f \cdot \mathbf{i} = f_x$. Also, the maximum rate of change of f is in the direction of ∇f. Why?

Heat Conduction Example (Boyce and DiPrima): The temperature distribution in a thin rectangular plate is given by

$$u(x,y) = \frac{60}{\sinh\frac{\pi}{2}} \sin x \sinh y + \frac{20}{\sinh \pi} \sin 2x \sinh 2y. \tag{3}$$

The plate is defined by the region R: $0 \le x \le \pi$, $0 \le y \le \pi/2$.

- Give the temperature on the edges of the plate.
- At the middle of the plate, in what direction is the temperature increasing most rapidly and what is this rate of change. In what direction is the heat flux vector increasing most rapidly.
- Compute the flux vector $\vec{\phi}$ along the boundary of R. Assume that the thermal conductivity $K_0 = 1$. Note: you need four different expressions.

Green's Theorem. Green's Theorem gives a relationship between line integrals along a nice closed curve C and the area integral over the region R enclosed by C. If $P(x,y)$ and $Q(x,y)$ are smooth functions then

$$\oint_C P dx + Q dy = \int\int_R (Q_x - P_y) dx dy. \tag{4}$$

An alternate form of (4), which is particularly useful for us, is

$$\oint_C \nabla u \cdot \mathbf{n}\, ds = \int\int_R \nabla^2 u \, dx dy, \tag{5}$$

where $\mathbf{n}$ is the outward normal to region R.

- Try to derive (5) by letting $Q = u_x$ and $P = -u_y$ in (4) and then showing that the line integral reduces to the line integral in (5).

- Compute the total heat flux leaving the plate, i.e. region R, for the heat conduction example of above. The temperature $u(x, y)$ is given by (3). Calculate both the line and the area integrals. Does your result make physical sense? Explain!

Eigenvalue Problem. The boundary value problem that arises in SOV is also called a Sturm-Liouville eigenvalue problem and is similar to the eigenvalue problem for a matrix $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 1 \end{pmatrix}.$$

- State the eigenvalue problem for $\mathbf{A}$ with an equation and in words.

- Compute the eigenvalues and eigenvectors.

- Compute the normalized eigenvectors, i.e. $\|u\|_2 = 1$

- Are the eigenvectors orthogonal?