

Math 481 Sample Questions - Exam 2

1. Consider the vibrating membrane problem

$$\begin{aligned}u_{tt} &= c^2 \nabla^2 u, 0 < x < \pi, 0 < y < 1, \\u(x, y, 0) &= 0, u_t(x, y, 0) = \sin \pi y \\u(0, y, t) &= u(\pi, y, t) = u(x, 0, t) = u(x, 1, t) = 0\end{aligned}$$

- Use separation of variables in time and state the Helmholtz equation BVP for the problem.
- Find the eigenvalues and eigenfunctions of the problem in (a).
- Construct $u(x, y, t)$ completely.

2. Solve the heat conduction problem

$$\begin{aligned}u_t &= \nabla^2 u, 0 < x < \pi, 0 < y < 1, \\u(x, y, 0) &= \sin \pi y \\u(0, y, t) &= u(\pi, y, t) = u(x, 0, t) = u(x, 1, t) = 0\end{aligned}$$

3. Determine the eigenvalues and their corresponding eigenfunctions for the Helmholtz BVP:

$$\begin{aligned}\nabla^2 \phi + \lambda \phi &= 0 \text{ in } R \\ \phi &= 0 \text{ on } C\end{aligned}$$

where the region R consisting of the x -axis from -1 to 1 and the portion of the circle $x^2 + y^2 = 1$ in the upper half-plane. The boundary of R is C . How does this problem differ from a Sturm-Liouville problem?

4. Formulate carefully and solve the heat conduction problem

$$\begin{aligned}u_t &= \nabla^2 u, \text{ in the unit circle} \\u(r, \theta, 0) &= f(r) \\u(1, \theta, t) &= 0.\end{aligned}$$

5. Assume $G(x, y; x_0, y_0)$ is the Green's function associated with $\nabla^2 u = g(x, y)$ in the unit square subject to the boundary conditions:

$$u(0, y) = 0, u(1, y) = f(y), u(x, 0) = 0, u(x, 1) = 0.$$

- State the problem for G . How would you construct G ? Do not solve.
- Express the solution u in terms of G .

6. Consider

$$u''(x) = f(x) \text{ with } u(0) = 0 \text{ and } u'(1) = 0.$$

- Formulate the problem for the Green's function G using the delta function.
- Construct G two ways - using a jump condition and eigenfunction expansion.
- Write the solution $u(x)$ of the above equation in terms of G . If $f(x) = x^2$, give the explicit form for u using G and compare it with a direct integration.
- If the boundary condition at $x = 1$ is replaced by $u'(1) = A$, write the solution in terms of G .

7. Consider the heat conduction problem

$$\begin{aligned}u_t &= u_{xx} + e^{-t} \sin 3x, \quad 0 < x < \pi, \quad t > 0, \\u(x, 0) &= 0 \\u(0, t) &= 0, \quad u(\pi, t) = 1\end{aligned}$$

- Solve by converting the nonhomogeneous bc's into homogeneous bc's.
 - Solve without converting the boundary conditions into homogeneous bc's.
 - Compare the two results.
8. The free-space Green's function for Laplace's equation in two dimensions is $G(x, y; x_0, y_0) = \frac{1}{4\pi} \ln[(x-x_0)^2 + (y-y_0)^2]$. Determine the Green's function for $y > 0$, $\nabla^2 G = \delta(x-x_0)\delta(y-y_0)$ subject to $\partial G/\partial y = 0$ on $y = 0$. Use the result to solve $\nabla^2 u = 0$ with $\partial u/\partial y = h(x)$ on $y = 0$.
9. Consider the linear operator $L[\psi] \equiv \psi'' + \psi$, $0 < x < \pi$, with the homogeneous boundary conditions $\psi(0) = 0$, $\psi(\pi) = 0$.
- Find the eigenvalues and eigenfunctions of $L[\psi] + \lambda\psi = 0$ with the associated boundary conditions.
 - Does the problem $L[u] = f(x)$ with $u(0) = 0$, $u(\pi) = 0$ have a unique solution? Explain.
 - For what values of β does $L[u] = x - \beta$ with $u(0) = 0$, $u(\pi) = 0$ have a solution, construct it.

10. Consider

$$u_t = ku_{xx} - u_x, \quad -\infty < x < \infty, \quad u(x, 0) = f(x).$$

- Find the Fourier transform $U(\omega, t)$ of the solution.
- Try to invert U .