Math 584 - Applied Stochastic Models - C. Tier - Fall 2005

Homework - 1

1. If X is a non-negative, integer-valued random variable then prove that
$$\text{E}[X] = \sum_{k=0}^{\infty} \text{Pr}[X > k].$$

2. Let Y is a random variable with a binomial distribution, i.e. $\text{Pr}[Y = k] = \binom{n}{k} p^k q^{n-k}, k = 0, \ldots n,\ p > 0,\ p + q = 1$.

   (a) Derive the probability generating function (pgf) of Y.

   (b) Use the pgf to find E$[Y]$ and Var(Y).

3. Consider a sequence of Bernoulli trials with p equal to the probability of success. Also, let T be the number of trials until the first success.

   (a) Find the probability distribution of T. What type of distribution is it?

   (b) Verify that your result in a. is normalized.

   (c) Compute the pgf of T and use it to compute the mean time until the first success.

4. (Karlin) What is the probability that $ax^2 + bx + c$ has real roots if a, b, c are uniform random variables on (0,1).

5. Let X and Y be two independent, non-negative, integer-valued random variables with probabilities
$$\text{Pr}[X = j] = a_j, \quad \text{Pr}[Y = k] = b_k.$$
Find the pdf of $S = X + Y$ and derive Pr$[S = i]$.

6. (Karlin) Let X be a non-negative random variable and let
$$Y = \min\{X, c\} = \begin{cases} X, & X \le c, \\ c, & X > c. \end{cases}$$
Find E$[Y]$ in terms of the PDF of X.

7. Let X be a Poisson random variable with parameter λ. Suppose the λ is an exponential random variable with mean $1/c$. Compute Pr$[X = k]$.

8. Assume T_1 and T_2 are independent, identically distributed (i.i.d) exponential random variables with pdf $p(t) = \lambda e^{-\lambda t}$. Derive the probability distribution function (PDF) of $T = T_1 + T_2$.

9. X_j are i.i.d Bernoulli random variables with $\Pr[X_j = 0] = q$ and $\Pr[X_j = 1] = p$, $p + q = 1$. The sum of a random number N of these random variables is denoted by

$$S_N = X_1 + X_2 + \cdots + X_N,$$

where N has a Poisson distribution with parameter μ. Derive (a) $\Pr[S_N = k]$, the generating function for S_N, and (c) $\mathrm{E}[S_N]$.

10. Let T be the time of the first success in a sequence of Bernoulli trials (see earlier problem) with $p_n = \Pr[T = n]$. Suppose that the time between Bernoulli trials is re-scaled to be ε instead of 1 and the probability of success is also scaled as $p = \lambda\varepsilon$. Determine the limit of p_n as $\varepsilon \to 0$. State the density function of the new continuous random variable.