Math 584 - Applied Stochastic Models - C. Tier - Fall 2005

Homework - 2

1. Consider a discrete-state Markov chain with transition matrix

$$A = \begin{pmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{pmatrix}$$

   (a) Find the equilibrium state vector $\vec{\pi}$ directly, if it exists.

   (b) Find an explicit form for A^n.

   (c) Is it true that $\lim_{n\to\infty} \vec{p}(0)A^n = \vec{\pi}$?

2. Let A be an $n \times n$ stochastic matrix. Show that the eigenvalue problems $wA = \lambda w$ and $Ax = \lambda x$ have the same eigenvalues but it not in general true that $w^T = x$. For what type of matrix A are the eigenvectors transposes of each other?

3. A Markov chain with state-space {0,1,2} has transition matrix

$$A = \begin{pmatrix} .4 & .2 & .4 \\ .6 & 0 & .4 \\ .2 & .5 & .3 \end{pmatrix}$$

   (a) Sketch the transition diagram.

   (b) Based on the transition diagram can you conclude that A is irreducible and aperiodic. You might use the Perron-Frobenius theory to make the same conclusion.

   (c) Use Matlab or Maple to compute A^{15}.

   (d) Find the equilibrium state vector $\vec{\pi}$ directly, if it exists. Is the result consistent with (c)?

4. Let $X(n)$ be the location of a random walk on the state-space $S = \{0, 1, \dots, N\}$. The state 0 is absorbing and the state N is reflecting. At the states $1, \dots, N-1$, the random walk can jump left with probability l or jump right with probability r, where $r + l = 1$.

   (a) Draw the transition diagram and state the transition matrix A.

   (b) Find the equilibrium solution, if possible.

   (c) Let $u_k = \Pr[X(n^*) = 0 | X(0) = k]$. Derive a problem for u_k involving the backward equation and solve it. Here n^* is the first time (first passage time) that X reaches 0.

   (d) Let $T_k = \mathrm{E}[n^* | X(0) = k]$. Derive a problem for T_k involving the backward equation and solve it. Here T_k is referred to as the mean first passage time.