Analyzing Nonlinear Systems of ODEs - Charles Tier

Nonlinear differential equations often arise in modelling biological systems. When the models are characterized by quantities that interact then systems of equations need to be analyzed. Below is brief introduction to the analysis of nonlinear systems of ODEs.

Suppose we consider a system of two *autonomous* first-order equations

$$\begin{aligned}\frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y).\end{aligned} \tag{1}$$

The equations are *autonomous* since f and g are independent of t. A solution of (1) is a pair of two functions $x(t)$ and $y(t)$ that satisfy the equations plus any given initial conditions. It is rare that an analytic formula for the solution of (1) can be constructed so we usually must construct a numerical solution or study the qualitative behavior of the solution. Maple can be used to do both using the DEplot command. A qualitative analysis proves extremely helpful for constructing numerical solutions. In either case, we find it convenient to study the solutions in the *phase plane* (x, y). We note the following properties (1):

1. The solution pair $(x(t), y(t))$ represents a parametric set of equations in the *phase plane* with t as a parameter.
2. The vector $(\frac{dx}{dt}, \frac{dy}{dt})$ is tangent to the solution curve defined by $(x(t), y(t))$.

Direction Field: We can use fact 2 above to generate a direction field for (1) either roughly by hand or using the DEplot command in Maple. A graph of a family of solutions is called a *phase portrait* and can be generated from the direction field or numerically using the DEplot command. A disadvantage of both the numerical or direction field approach is that the parameters in (1) need to be specified.

Stability Analysis: Special solutions of (1) play a central role in the behavior of the solutions in the *phase plane*. The equilibrium, steady-state or critical points are constant solutions, $x(t) = X$ and $y(t) = Y$, which satisfy the nonlinear system of equations $f = 0$ and $g = 0$. A systematic approach to stability analysis is summarized below.

1. Locate the critical points (X, Y) by solving:

$$\begin{aligned}f(X,Y) &= 0 \\ g(X,Y) &= 0\end{aligned}$$

2. Local Analysis near (X, Y). We study (1) locally near each rest point. This leads to a linear approximation to (1), which we can solve solve explicitly. If the original nonlinear system is *almost linear* we can determine the stability of the critical point of (1) based on the behavior of the linear one. The system (1) is *almost linear* if f and g are twice differentiable. The procedure is as follows.

We replace x and y by

$$\begin{aligned}x(t) &= X + \xi(t) \\ y(t) &= Y + \eta(t)\end{aligned} \tag{2}$$

where $\xi(t)$ and $\eta(t)$ are small. Next we substitute (2) into (1) to obtain

$$\begin{aligned} \frac{d\xi}{dt} &= f(X+\xi(t), Y+\eta(t)) &\approx \frac{\partial f}{\partial x}\xi + \frac{\partial f}{\partial y}\eta \\ \frac{d\eta}{dt} &= g(X+\xi(t), Y+\eta(t)) &\approx \frac{\partial g}{\partial x}\xi + \frac{\partial g}{\partial y}\eta. \end{aligned} \tag{3}$$

Here the partial derivatives are evaluated at the critical point (X, Y). We can write (3) in matrix form as

$$\frac{d\vec{\zeta}}{dt} = \mathbf{J}(X,Y)\,\vec{\zeta}, \quad \vec{\zeta} = \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \tag{4}$$

where the matrix $\mathbf{J}(X, Y)$ is called the *Jacobian* matrix (community matrix in ecology) defined by

$$\mathbf{J}(X,Y) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(X,Y)}.$$

3. Eigenvalues of $\mathbf{J}$. The stability of the critical point (X, Y) of (1) can determined by studying the eigenvalues of $\mathbf{J}$, except for special cases. This is analogous to studying the first derivative that appears in the linearized equation for a single nonlinear equation. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{J}$. Both the *Jacobian* and the eigenvalues can be easily obtained using Maple. The classification of the critical point is summarized below.

- λ_1 and λ_2 are real
  1. $\lambda_1 > 0$ and $\lambda_2 > 0$ - (X, Y) is unstable node
  2. λ_1 and λ_2 are of opposite sign - (X, Y) is unstable (saddle point)
  3. $\lambda_1 < 0$ and $\lambda_2 < 0$ - (X, Y) is asymptotically stable node
- λ_1 and λ_2 are complex - $\lambda = a \pm i\,b$
  1. $a < 0$ - (X, Y) is asymptotically stable (spiral)
  2. $a > 0$ - (X, Y) is unstable (spiral)
  3. $a = 0$ - test is inconclusive (The linear system (3) has a center at (0,0) with closed solution trajectories around it.)
- $\lambda_1 = \lambda_2$ (real)
  1. one independent eigenvector
     (a) $\lambda_1 > 0$ - (X, Y) is unstable node or spiral ($(0, 0)$ is an unstable improper node of (3)).
     (b) $\lambda_1 < 0$ - (X, Y) is an asymptotically stable node or spiral ($(0, 0)$ is an asymptotically stable improper node of (3)).
  2. two independent eigenvectors
     (a) $\lambda_1 > 0$ - (X, Y) is an unstable node or spiral ($(0, 0)$ is an unstable proper node of (3)).
     (b) $\lambda_1 < 0$ - (X, Y) is an asymptotically stable node or spiral ($(0, 0)$ is an asymptotically stable proper node of (3).