MATH 504 PROBLEM SET 5

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Problem 1. Let S be a stationary set in a regular cardinal λ. Show that exactly one of the following holds:

(a) There is a stationary set $S' \subseteq S$ such that for every $\alpha \in S'$, α is regular.

(b) There is a stationary set $S' \subseteq S$ and $mu < \lambda$ such that for every $\alpha \in S$, $cf(\alpha) = \mu$.

Moreover prove that in case (1), $\{\alpha \in S' \mid S \cap \alpha \text{ is not stationary in } \alpha\}$ is stationary. [Hint: Let C be a club, consider $\min(S \cap \lim(C))$.]

Problem 2. A cardinal κ is called a *Mahlo Cardinal* is $\{\alpha < \kappa \mid \alpha \text{ is regular}\}$ is stationary in κ.

(a) Prove that if κ is Mahlo, then $\{\alpha < \kappa \mid \alpha \text{ is a strongly inaccessible cardinal}\}$ is a stationary.

(b) Conclude that if κ is Mahlo then κ is the κ^{th}-inaccessible.

(c) Conclude that the first inaccessible is not Mahlo.

Problem 3. Prove the other direction of Födor's Lemma: If $A \subseteq \kappa$ is a set such that for every regressive function $f : A\kappa$ there is an unbounded $B \subseteq A$ such that $f \restriction B$ is constant, then A is stationary [Hint: Assume toward a contradiction that C is a club disjoint from A and define $f : A \to \kappa$ by $f(\alpha) = \max(\alpha \cap C)$, this is a regressive function!]

Problem 4. Prove that if $X, Y \in V_\alpha$, then:

(a) $X \cap Y, X \cup \alpha, X \setminus \alpha, X \Delta Y \in V_\alpha$ (Since the sup is taken over less elements so $Rank(X \cap Y), ... \leq Rank(X)$), and in general, if $Z \subseteq X$ then $Rank(Z) \leq Rank(X)$.

(b) $P(X) \in V_{\alpha+1}$ (Since $Rank(P(X)) = \sup(Rank(Y) + 1 \mid Y \subseteq X) = \alpha + 1$.

(c) $X \times Y \in V_{\alpha+2}$. (Since for every $x \in X, y \in Y$, $Rank(\langle x, y\rangle) \leq \alpha$, then $X \times Y \subseteq V_{\alpha+1}$ and therefore $X \times Y \in V_{\alpha+2}$.

(d) If E is a relation on X, then $E \in V_{\alpha+2}$ (Since $E \subseteq X \times Y$).

(e) If E is an equivalence relation, for $a \in X$, $[a]_E \in V_\alpha$.

(f) Also $X/E \in V_{\alpha+1}$.

Problem 5. Let M be any class, prove the following:

(a) $M \models Ax0$ iff $M \neq 0$.

(b) If M is a transitive class, then $M \models$extensionality.

(c) If $\forall x, y \in M \exists z \in M.\{x, y\} \subseteq z$ then $M \models$pairing.

(d) If $\forall x \in M \exists y \in M. \cup x \subseteq y$, then $M \models$ union.

[Recall that $M \models \phi$ is the statement ϕ^M.]

Problem 6. Suppose that M is a class such that for every x, if $x \subseteq M$ then $x \in M$. Prove that $WF \subseteq M$.[Hint, prove by induction that $V_\alpha \subseteq M$].

Problem 7. Write the relativization of the Power set Axiom, the Axiom of Choice, the Axiom of Foundation, and the Axiom of infinity to a class $M = M_\phi$.